

only use on me!!

Q(a) Primary Trig Ratios \rightarrow Sine CAH TOA

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

Secondary (Reciprocal) Trig Ratios

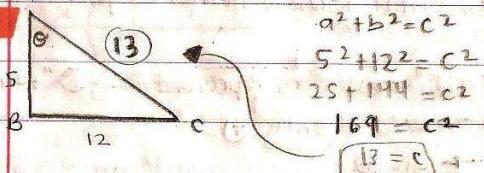
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} \quad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

These are just reciprocals of the ones listed directly above them.

TRICK: each "pair" needs to have a prefix "co". By looking at the original trig functions you know more naturally, compare and see which one needs "co" and which one doesn't.

sine \rightarrow cosecant cosine \rightarrow secant tangent \rightarrow cotangent

Q(b) Finding the secondary trig ratios from a right triangle



$$a^2 + b^2 = c^2$$

$$5^2 + 12^2 = c^2$$

$$25 + 144 = c^2$$

$$169 = c^2$$

$$13 = c$$

① You need all sides in order to use any of the trig ratios. (if using ALL ratios) So, find the hypotenuse by using $a^2 + b^2 = c^2$.

② Using θ , create ratios for each secondary trig function.

NOTE: when finding the ratio, if you have a radical in the denominator, rationalize to simplify.

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} \quad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

$$\text{hyp} = 13 \\ \text{opp} = 12$$

$$\text{hyp} = 13 \\ \text{adj} = 5$$

$$\text{adj} = 5 \\ \text{opp} = 12$$

$$\csc\theta = \frac{13}{12}$$

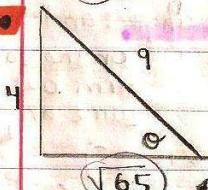
$$\sec\theta = \frac{13}{5}$$

$$\cot\theta = \frac{5}{12}$$

or reduce fractions if possible!

*Use steps
example

① for this
question
as well! :)



$$a^2 + b^2 = c^2 \\ 4^2 + b^2 = 9^2 \\ b^2 = 81 - 16 \\ b^2 = 65 \\ b = \sqrt{65}$$

simplify if possible!

$$\csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}}$$

$$\csc\theta = \frac{9}{4}$$

$$\sec\theta = \frac{9}{4} \cdot \frac{\sqrt{65}}{\sqrt{65}} = \frac{9\sqrt{65}}{65}$$

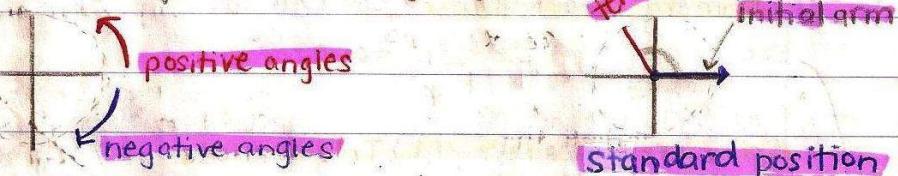
$$\cot\theta = \frac{\sqrt{65}}{4}$$

$$\sec\theta = \frac{9\sqrt{65}}{65}$$

NOTE: When finding co-terminal angles using a calculator to add/subtract, keep the negative sign!!

terminal arm is the arm that rotates around the origin.

(2c) Definitions / Diagrams for rotating angles



positive angles rotate counter-clockwise.
negative angles rotate clockwise.

Standard position aka:
the choice of using
positive x-axis as 0° .

standard position

An angle in standard position has its vertex @
the origin and its initial arm on
the positive x-axis.

Distance measured counter-clockwise
from the initial arm to your
terminal arm = angle.

Labelling Quadrants

- Quadrants in trigonometry are labelled using letters to help identify which ratios are \oplus/\ominus in which quadrant.
- Quadrants are literally labelled with Roman numerals just like a cartesian plane.

II	I
S $180^\circ \dots$ $-180^\circ \dots$	A $0^\circ, 360^\circ \dots$ $0^\circ, -360^\circ \dots$
III	IV
$270^\circ \dots$ $-90^\circ \dots$	C T

"CAST" → letters represent "only \oplus " ratios

C → cosine only \oplus

A → all \oplus

S → sine only \oplus

T → tangent only \oplus

Quadrantal angles are the angle measures along each of the quadrant "dividers" and can be labelled an infinite amount of times.

Negative quadrantal angles also exist for negative rotation angles. These can also be labelled infinitely.

Reference/Related Acute Angle



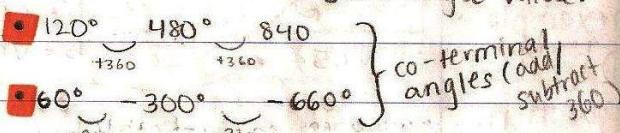
To find this related acute angle, ALWAYS draw your right triangle towards the x-axis!!

think about this to remember:

always draw to x-axis!

Co-terminal angles: angles that fall

on the same terminal arm but are not the same angle value.



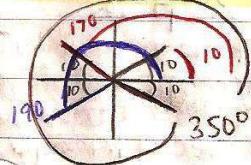
Principal angle: Angle within first positive revolution. It is also the least positive co-terminal angle.

$[0, 360)$

To write an equivalent trig ratio, you need to look at your original angle and find its related acute angle. With this, you can find ratios that have the same magnitude. BUT, to make it truly equivalent, signs have to be the same. So, look for where your equivalent ratio will match your original using CAST.

Extra Notes → Rotation Angles

- Ratio values are same in magnitude (not in sign) for angles that share a related acute angle.



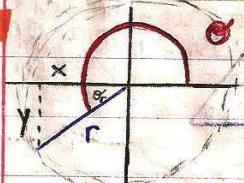
There are 4 angles. All have a related acute angle of 10° .
Each has a different principal angle.
 ① 10° ③ 190°
 ② 170° ④ 350°

If you took the trig ratio of these angles given, you would see that all their ratios are the SAME! (in magnitude, not in sign)

(using sine)
for example...

$$\begin{array}{ll} \text{① } \sin 10^\circ = 0.17... & \text{③ } \sin 190^\circ = -0.17... \\ \text{② } \sin 170^\circ = 0.17... & \text{④ } \sin 350^\circ = -0.17... \end{array}$$

- How can you use SOH CAH TOA on non-acute angles?



- The related acute angle can be labelled θ .
- The opposite can be labelled "y" because it's y units long. (y is subject to being $\pm\theta$)
- The adjacent can be labelled " x " because it's x units long. (x is subject to being $\pm\theta$)

- The terminal arm can be labelled "r" for radius. It is the hypotenuse in our right angle triangle.

Before you can assign any of these variables, you need to create a related acute angle by drawing a right angle triangle TOWARDS the x-axis.

INSTEAD OF SOH CAH TOA...

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos \theta = \frac{x}{r}$$

slope of
terminal arm
(r)

$$\tan \theta = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan \theta = \frac{y}{x}$$

In these ratios:

- radius is never negative
- related acute angle is never negative
- principal angle is never negative.

RECIPROCAL VERSIONS:

$$\csc \theta = \frac{r}{y}$$

$$\sec \theta = \frac{r}{x}$$

$$\cot \theta = \frac{x}{y}$$

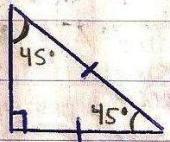
If y is drawn upwards \rightarrow θ
If y is drawn downwards \rightarrow θ

If x is drawn to the right \rightarrow θ

If x is drawn to the left \rightarrow θ

①(a) Special Triangles

" $45^\circ, 45^\circ, 90^\circ$ "



This is an isosceles special triangle. Because it has 2 sides equal and a 90° angle the other two must be 45° .

$$\rightarrow 180^\circ - 90^\circ = 90^\circ$$

$$90^\circ \div 2 = 45^\circ \text{ each}$$

" $30^\circ, 60^\circ, 90^\circ$ "



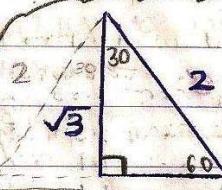
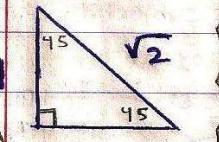
This is an equilateral triangle. Half of an equilateral triangle is a special triangle.

Within this equilateral triangle, all angles are 60° because:

$$60^\circ + 60^\circ + 60^\circ = 180^\circ$$

When cut in half, a 30° , and 90° are "produced".

The Pythagorean Theory on any triangle with this pattern follows this special triangle:



The Pythagorean Theory on any triangle with this pattern follows this special triangle.

all $30-60-90$ triangles have this ratio.

WHERE DID $1, \sqrt{3}, 2$ come from?

$$a^2 + b^2 = c^2$$

We created an equilateral triangle with an easy number to "split" up for our side "ratios". So, 2 became our c and 1 became our b .

$$a^2 + (1)^2 = (2)^2$$

$$a^2 + 1 = 4$$

$$\sqrt{a^2} = \sqrt{3}$$

$$a = \sqrt{3}$$

①(b)

WHERE DID $1, 1, \sqrt{2}$ come from?

$$a^2 + b^2 = c^2$$

We know a, b are both equal. To create an easy "ratio" on each side, choose 1 as your a, b .

$$(1)^2 + (1)^2 = c^2$$

$$1+1=c^2$$

$$\sqrt{2}=c^2$$

$$\sqrt{2}=c$$

Think about K-factor + similar Δ 's

To use these triangles to find missing side lengths, multiply your side lengths by the "scale factor" (how many more times a length is larger/smaller than another)

How to remember which side is across what angle?

The sequence of side lengths and angles from least to greatest correspond to each other. The corresponding side length in each sequence is across from the angle.

$$A: 45^\circ, 45^\circ, 90^\circ$$

$$S: 1, 1, \sqrt{2}$$

Both 45° angles are across from side lengths

1 and 1. The 90° angle is across from the $\sqrt{2}$ side length.

$$A: 30^\circ, 60^\circ, 90^\circ$$

$$S: 1, \sqrt{3}, 2$$

The 30° angle is across from side length 1.

The 60° angle is across from side length $\sqrt{3}$.

The 90° angle is across from side length 2.

- ③(a) "CAST" is used to identify which trig ratios are positive in which quadrant. Note: CAST starts at quadrant IV not I.
For details on CAST, see journal #②(c) Labeling Quadrants.
Another way to help you remember is "All Students Take Calculus".
This method starts at quadrant I.

① (3b)

290° is a rotation angle.

NOTE: all rotation angles must be attached to the origin.

② Draw a right angle triangle from your point.

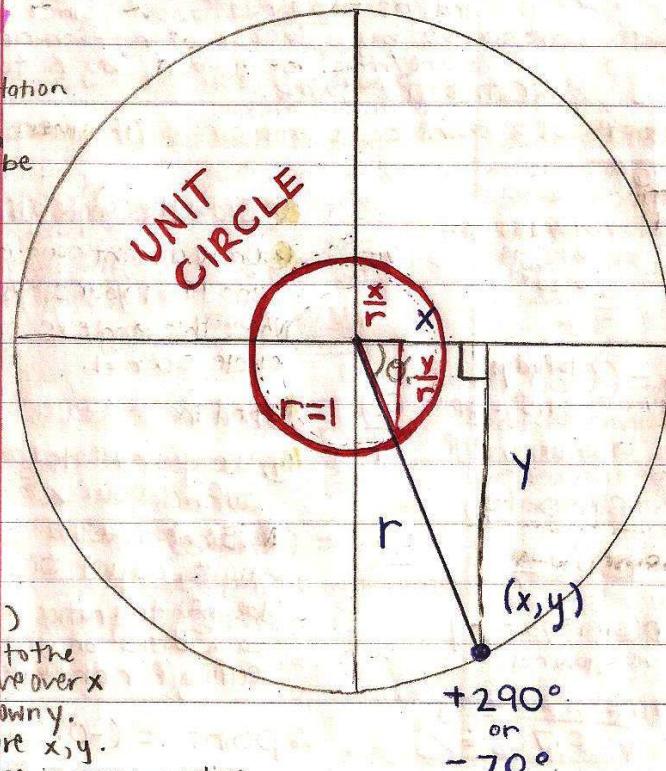
NOTE: draw it hugging/ connecting to the x-axis.

③ The point on the terminal arm can be labelled (x, y)

because to get to the point you move over x units and down y.

so, your legs are x, y .

Your hypotenuse is your r , radius.



NEW TRIG DEFINITIONS

(This is a "simulation" explaining what a unit circle is.)

VARIABLE NOTES

$x \rightarrow$ left = negative
right = positive

$y \rightarrow$ down = negative
up = positive

$r \rightarrow$ always positive
 θ (related acute angle)

\angle always positive
(that's why you have to adjust $\pm x, y$ when solving for θ)

If you put a non-acute angle into a trig ratio, you don't have to adjust the negatives. If you use its related acute angle, then you **MUST** adjust the negatives. (Refer to ③d)

④ Using knowledge of SOH CAH TOA, what would the new trig definitions be?

(Look at/refer to "Extra Notes → Rotation Angles")

Using x, y, r

$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

(To do this, simply replace "opp, adj, hyp" with their corresponding variables according to the circle drawn on the previous page.)

⑤ Draw a smaller circle and a smaller right-angle triangle by making $r=1$. This scaled-down triangle = **Unit Circle**. Because the triangle was scaled down to $r=1$, the scale factor (k -value) is r . So, the sides x and y become $\frac{x}{r}$ and $\frac{y}{r}$.

③(a) What do the sides on the unit circle triangle represent?

hypotenuse $\rightarrow r \rightarrow 1$

leg $\rightarrow x \rightarrow \frac{x}{r}$

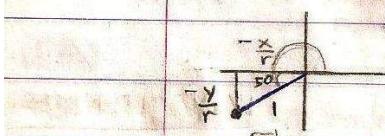
leg $\rightarrow y \rightarrow \frac{y}{r}$

A unit circle triangle takes trig ratios and simplifies/scales them to the simplest form. This form is when $r=1$.

And, as explained above, since all sides are scaled by a factor of r , each original x and y side are/r must be divided by r to keep ratios the same.

③(d) Find co-ordinates of a point on a unit circle (if given angle of rotation)

angle = 230°



$$\text{related acute: } 230^\circ - 180^\circ = 50^\circ$$

$$\text{to find } x: \cos 50^\circ = \frac{-x}{r} \quad \text{hyp.}$$

$$\text{to find } y: \sin 50^\circ = \frac{-y}{r} \quad \text{hyp.}$$

$$\cos 50^\circ = \frac{-x}{r}$$

$$\sin 50^\circ = \frac{-y}{r}$$

$$\cos 50^\circ = -x$$

$$\sin 50^\circ = -y$$

$$r(\cos 50^\circ) = -x$$

$$r(\sin 50^\circ) = -y$$

we know $r=1$

$$\cos 50^\circ = -x$$

$$\sin 50^\circ = -y$$

$$-0.643 = x$$

$$\sqrt{-0.766} = y$$

① Draw the angle given

② Draw a right-angle triangle and label your sides.

Note: this angle IS on a unit circle so $r=1$.

③ Find the related acute angle

④ Use your related acute angle to find your x and y .

NOTE: when using related acute angle, adjust signs of x and y .

Identify trig ratios to use for each based on what you are solving for and what you know (r).

$$\therefore \text{point} = (-0.643, -0.766)$$

CHECK USING $x^2 + y^2 = r^2$

If you are given a ratio where you have to do an inverse to solve for the related acute, **DROP ALL negatives** for your triangle measures because $r = \text{positive!}$

(3e) find coordinates of a point on a NON-unit circle (given

$$\bullet \text{angle} = -325^\circ$$

$$\text{radius} = 5$$

$$\text{related acute} =$$

$$360 - 325$$

$$= 35^\circ$$

to find x:

$$\cos 35 = \frac{\text{adj}}{\text{hyp}} = \frac{x}{5}$$

$$\cos 35 = \frac{x}{5}$$

$$5(\cos 35) = x$$

$$4.096 \doteq x$$

check: using $x^2 + y^2 = r^2$

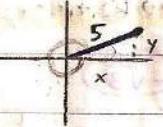
to find y:

$$\sin 35 = \frac{\text{opp}}{\text{hyp}} = \frac{y}{5}$$

$$\sin 35 = \frac{y}{5}$$

$$5(\sin 35) = y$$

$$2.868 \doteq y$$



① Draw the angle you are given (radius & angle of rotation)

② Draw the right-angle triangle and label your sides.

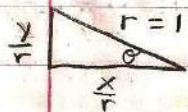
③ Find the related acute angle

④ Use your related acute angle to find x and y by using appropriate trig ratios. Adjust negatives (not necessary in this example).

$$\therefore \text{point} = (4.096, 2.868)$$

TRICK → finding x and y in a unit/non-unit circle

unit circle



to find y
r is given.

So, you know r.
you are finding y
which is opposite to θ . So, you use sine.

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin \theta = \frac{y}{r}$$

$$\sin \theta = \frac{y}{1}$$

$$\sin \theta = \frac{y}{r}$$

$$\sin \theta = \frac{y}{1}$$

$$r(\sin \theta) = y$$

we know $r=1$ in this case

$$\sin \theta = y$$

$$\cos \theta = \frac{x}{r}$$

$$\cos \theta = \frac{x}{1}$$

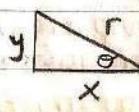
$$\cos \theta = x$$

$$r(\cos \theta) = x$$

we know $r=1$ in this case

$$\cos \theta = x$$

non-unit circle



to find y

you are finding y
this is opposite to θ . r is the hypotenuse. so, use sine.

so, use cosine.

$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

$$r(\sin \theta) = y$$

r can be anything
in this case so we leave it as a variable, r.

* WORKING IN DEGREES *

(5a) Find output ratio values given angles (given θ)

• Value of a trig ratio for a special triangle

$$\sin(-60^\circ)$$

$$\theta_r = 60^\circ$$

$$x=1$$

$$y=-\sqrt{3}$$

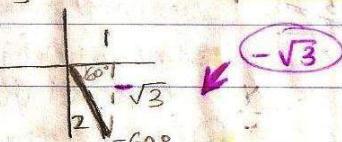
$$r=2$$

$$\sin \theta = \frac{y}{r}$$

$$\sin(-60^\circ) = \frac{y}{r} = \frac{-\sqrt{3}}{2}$$

$$\sin(-60^\circ) = -\frac{\sqrt{3}}{2}$$

this is your output ratio
Value for $\sin(-60^\circ)$



① Draw a picture of your given angle (draw to scale as best as possible)

② Find the related acute angle of the triangle

* We find the related acute to determine if special or not.
Because $\theta_r = 60^\circ$, use the $30-60-90$ special triangle.

③ Label the sides of your triangle and fill in any negatives as needed.

④ Identify the ratio of sine ratio and fill in its trig definition.

• Value of a trig ratio for a quadrantal angle

$$\tan 270^\circ$$

point $(0, -1)$

$$x=0$$

$$y=-1$$

$$r=1$$

$$\tan \theta = \frac{y}{x} \quad x=(-1)$$

$$\tan 270^\circ = \frac{1}{0} = \text{undefined!}$$

$$\tan 270^\circ = \text{undefined}$$

The tangent ratio can be considered to be slope because it is $\frac{y}{x}$ or $\frac{\text{rise}}{\text{run}}$.

We can see our line for $\tan 270^\circ$ is a vertical line so it makes sense that $\tan 270^\circ$ is undefined.

① Draw a picture of your given angle (draw to scale as best as possible)

* This is quadrantal!

② Because you can't find a related acute angle in this case, use your unit circle! Label your point using the unit circle.

③ Now, because of your unit circle, you have x, y, r .

④ Identify the ratio of tangent and fill in its trig definition.

To find input angles of secondary ratios: take the inverse of the related primary trig ratio.

$$\csc \theta = \frac{5}{7}$$

in this example:
 $\csc \rightarrow \sin$
 $\text{so do } \sin^{-1}$

$$\theta_r = \sin^{-1}\left(\frac{5}{7}\right) \approx 46^\circ$$

or undefined
 $(\text{denom} = 0)$

(b) Find input angles given ratios

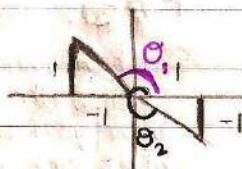
- If ratios are 0, +1, -1 (finding quadrantal angles)

$$\tan \theta = -1$$

$$\tan \theta = -\frac{1}{1} = \frac{y}{x} \quad \text{or} \quad \frac{1}{-1} = \frac{y}{x}$$

negative can be on
 x or y

$\tan \theta$
 in II and IV



- In this case, there are 2 input angles. Label them: θ_1 and θ_2 .

trap →
 2 negatives

- To find these angles, find your related acute angle.

$$\tan \theta_r = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{1}{-1} = -1$$

θ_r is $(+)$

$$\tan \theta_r = 1$$

$$\theta_r = \tan^{-1}(1)$$

$$\theta_r = 45^\circ$$

① All trig definitions are expressed as fractions.
 So, put -1 over 1 and state the tangent trig ratio.

② If y and x are both 1, we know the triangle must be special. ($45-45-90$)

③ Draw your triangle. NOTE: when drawing it, think about which quadrants tangent is negative in because that's where you draw your triangle.

you know that this is a special $45-45-90$ triangle. So, your related acute angle must be 45° .

OR

- To find θ_1 and θ_2 , you have to use your θ_r .

$$\theta_1 = 180^\circ - \theta_r = 180^\circ - 45^\circ = 135^\circ$$

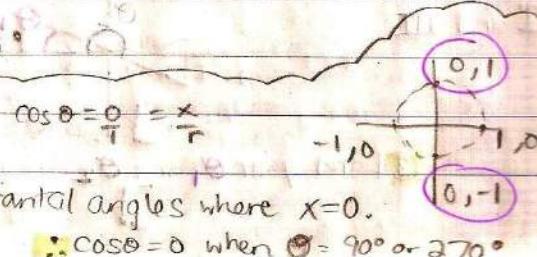
$$\theta_2 = 360^\circ - \theta_r = 360^\circ - 45^\circ = 315^\circ$$

$$\therefore \theta_1 = 135^\circ \quad \theta_2 = 315^\circ$$

note: if you had something like $\cos \theta = 0$:
 → put it over 1. This means $r=1$ (unit circle)

→ Using your unit circle, look at the quadrantal angles where $x=0$.

$\therefore \cos \theta = 0$ when $\theta = 90^\circ$ or 270°



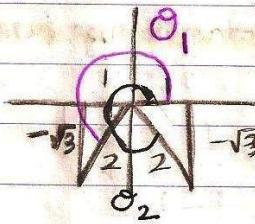
- If ratios have numbers from special triangles

$$\sin \theta = -\frac{\sqrt{3}}{2} = \frac{y}{r}$$

To draw: your θ_r is always across from your y . So, think about $-\sqrt{3}$ being your y when drawing.

In a $30-60-90$ triangle, your $\sqrt{3}$ is always across from 60° .

$$\therefore \theta_r = 60^\circ$$



- Find your related acute angle.

While we were drawing our special triangle, we noticed that our $-\sqrt{3}$ had to be across from our θ_r which we know is 60° .

- State the sine trig ratio.

② Notice that $r=2$. When $r=2$, we know we must have a $30-60-90$ triangle. Also note how in the previous example, the negative could be moved to the top/bottom, in this case it can't because $r \neq \theta$.

- Draw your triangle.

To know which quadrants to draw your triangle in, think of where sine is negative. Label θ_1 and θ_2

Long way of finding θ_r is: $\sin \theta_r = \frac{\sqrt{3}}{2} = \left(\frac{\text{opp}}{\text{hyp}}\right) = \left(\frac{y}{r}\right)$

$$\theta_r = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$$

- Use θ_r to solve for θ_1 and θ_2

$$\theta_1 = 180^\circ + \theta_r = 180^\circ + 60^\circ = 240^\circ$$

$$\theta_2 = 360^\circ - \theta_r = 360^\circ - 60^\circ = 300^\circ$$

$$\therefore \theta_1 = 240^\circ \quad \theta_2 = 300^\circ$$

- If ratios are neither special nor quadrantal

$$\cot \theta = 5.64 = \frac{x}{y} \quad (\text{OR}) \quad -5.64 = \frac{x}{y}$$

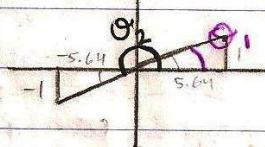
NOTE: both x and y can be negative

① This ratio value, even over a denominator of one, doesn't have any special traits to it. So, we need to find θ using a calculator.

(NOTE: the θ we find is our θ_r that we must use to find the input angles.)

\tan is ②:

I and III



- Label your θ_1 and θ_2

② Draw what we know from our trig definition. To draw, think: where is tangent positive?

④ Find the related acute angle

$$\cot \theta_r = \frac{5.64}{1} \Rightarrow \tan \theta_r = \frac{1}{5.64}$$

* we don't have a \cot^{-1} button, so we need to use \tan^{-1} .

$$\tan \theta_r = \frac{1}{5.64}$$

$$\theta_r = \tan^{-1} \left(\frac{1}{5.64} \right) \approx 10^\circ$$

⑤ After you have found your related acute angle, find your θ_1 and θ_2 .

$$\theta_1 = \theta_r \approx 10^\circ$$

$$\theta_2 = 180^\circ + \theta_r = 180^\circ + 10^\circ \approx 190^\circ$$

$$\therefore \theta_1 \approx 10^\circ \quad \theta_2 \approx 190^\circ$$

⑤c) state the general solution for the sequence of angular solutions

NOTE: θ (input angles) of trig ratios are infinite because each θ we find in the examples shown are strictly between 0° and 360° . All angle solutions are arithmetic sequences that can be written using this format:

$$\theta_n = \square^\circ + 360^\circ n \quad \text{OR} \quad \theta_n = \square^\circ + 180^\circ n$$

(using ⑤b) example 1)

$$\theta_1 = 135^\circ$$

$$\theta_2 = 315^\circ$$

these were input angles into a tangent ratio.
input angles in a tangent ratio have a
"constant difference" of 180° . So, you should

$$\text{use } \theta_n = \square^\circ + 180^\circ n$$

Only the sequence of general solutions for tangent follows this
because the signs of tangent ratios are not same in diagonal
quadrants. This creates a difference of 180° between solutions.

→ general: $\theta_n = 135^\circ + 180^\circ n \quad \{n \in \mathbb{Z}\}$ ← always define n as an
element of integers

(using ⑤b) example 2)

$$\theta_1 = 240^\circ$$

$$\theta_2 = 300^\circ$$

these were input angles into a sine ratio.
so, you need to state 2 different sets of general
solutions because the co-terminal angles of 240°
and 300° aren't related like those of the tangent ratio.

① $\theta_n = 240^\circ + 360^\circ n \quad \{n \in \mathbb{Z}\}$

② $\theta_n = 300^\circ + 360^\circ n \quad \{n \in \mathbb{Z}\}$

NOTE: Cosine input angles' general solutions are recorded
like sine's input angles.

Trigonometric Proofs (Identities)

(7a) What is the difference between an identity and equation?

An identity is a mathematical "statement" that is ALWAYS true.

An equation is only true for certain inputs.



$$LS = RS$$

$$\bullet ax + 5 = 1$$

$$\begin{aligned} 2x &= 1 - 5 \\ 2x &= -4 \\ \frac{2x}{2} &= \end{aligned}$$

$$x = -2$$

$$\bullet 2x + 1 = x + x + 1$$

$$\begin{aligned} 2x + 1 &= 2x + 1 \\ LS &= RS \end{aligned}$$

∴ identity because

$$LS = RS \text{ for ANY } x\text{-values}$$

∴ equation because

$$LS = RS \text{ only if }$$

$$x = -2$$

(7b) Trigonometric Identities

Reciprocal ID

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

Quotient ID

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean ID

$$(x^2 + y^2 = r^2)$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

(7c) Steps to follow when proving trig identities

① Start with complicated side

② Change all definitions/ratios to either sine or cosine using IDs ↑

③ Factor / expand

④ LCD / distribute the denominator

⑤ Multiply by the conjugate

(7d) Examples of trig proofs

$$\bullet \frac{\tan x + \tan y}{\cot x + \cot y} = \tan x + \tan y$$

$$\begin{aligned} LS &= \\ LCD &= \cos x \cos y \end{aligned}$$

$$\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}] \text{ QUOTIENT ID}$$

① Start with complicated side and change everything to sine/cosine.

$$\begin{aligned} LCD &= \sin x \sin y \\ &= \end{aligned}$$

$$\frac{\cos x}{\sin x} + \frac{\cos y}{\sin y}] \text{ QUOTIENT ID}$$

② Find LCD for fractions in numerator/denominator

$$LS = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}$$

$$\frac{\cos x \sin y + \cos y \sin x}{\sin x \sin y}$$

$$LS = \frac{(\sin x \cos y + \sin y \cos x) \cdot (\sin x \sin y)}{(\cos x \cos y) \cdot (\cos x \sin y + \cos y \sin x)}$$

(3) When dividing by a fraction, multiply by the reciprocal

(4) Cancel if possible

$$LS = \frac{\sin x \sin y}{\cos x \cos y}$$

(5) This isn't identical to your RS. Can you manipulate it so that it is??

$$LS = \frac{\sin x}{\cos x} \cdot \frac{\sin y}{\cos y}$$

$$\Rightarrow \tan x \tan y$$

$$RS = \tan x \tan y$$

$$\therefore LS = RS$$

(6) Write \therefore statement

$$\frac{\cos \theta}{1 - \sin \theta} = \sec \theta + \tan \theta$$

(1) The more complicated side looks like RS.

Change everything to sine/cosine on this side

$$RS = \sec \theta + \tan \theta$$

$$RS = \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}$$

$$RS = \frac{1 + \sin \theta}{\cos \theta}$$

OH NO! We are stuck
and our RS \neq LS

(2) Write out what we have so far to see if we can do anything more.

$$\frac{\cos \theta}{(1 - \sin \theta)} = \frac{(1 + \sin \theta)}{\cos \theta}$$

(3) The denominator of LS and numerator of RS are conjugates, so that may help us. Let's cross multiply.

$$\cos \theta (\cos \theta) = (1 - \sin \theta)(1 + \sin \theta)$$

$$\cos^2 \theta = 1 - \sin^2 \theta \quad \text{(diff of squares)}$$

pythagorean theory

$$\cos^2 \theta = \cos^2 \theta$$

$$\therefore LS = RS$$

(4) Write \therefore statement

$$\frac{1+\cos\theta + \sin\theta}{1+\cos\theta - \sin\theta} = \sec\theta + \tan\theta$$

LS =

$$(1+\cos\theta + \sin\theta) \cdot (1-\cos\theta - \sin\theta)$$

$$(1+\cos\theta - \sin\theta) \cdot (1-\cos\theta + \sin\theta)$$

$$\frac{1-\cos^2\theta + \sin\theta + \cos\theta - \cos^2\theta + \cos\theta \sin\theta}{1-\sin\theta - \sin^2\theta + \sin\theta + \cos\theta}$$

$$\frac{1-\cos^2\theta + \sin\theta + \cos\theta - \cos^2\theta + \cos\theta \sin\theta}{-1+\cos\theta + \cos\theta \sin\theta - \sin\theta}$$

$$\frac{1+\sin\theta - \cos^2\theta + \sin\theta + \sin^2\theta}{1-\cos^2\theta + \cos\theta \sin\theta + \cos\theta \sin\theta - \sin\theta}$$

$$\frac{\sin^2\theta + \sin\theta + \sin\theta + \sin^2\theta}{\sin^2\theta + \cos\theta \sin\theta + \cos\theta \sin\theta - \sin^2\theta}$$

$$\frac{2\sin^2\theta + 2\sin\theta}{2\cos\theta \sin\theta}$$

$$\frac{2\sin\theta(\sin\theta + 1)}{2\cos\theta \sin\theta}$$

$$2\cos\theta \sin\theta$$

$$\frac{\sin\theta + 1}{\cos\theta} \Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}$$

$$LS = \tan\theta + \sec\theta$$

$$\therefore LS = RS$$

① LS seems more complicated, so begin with that.

Notice how everything is already sine/cosine.

② Looking at what we can do, the only thing it looks like we can do is multiply by the conjugate.

③ FOIL numerator and denominator. BE CAREFUL

④ Look for cancellations because a lot of addition/subtraction is taking place.

⑤ Use any identities to simplify and CLT

⑥ common factor numerator then cancel because of multiplication

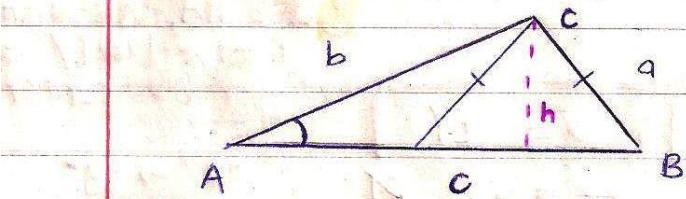
⑦ This doesn't look exactly like RS. Can you manipulate it so that it does?

⑧ Write ∴ statement

(8a) Ambiguous case → WHAT IS IT?

What is it? The ambiguous case is when we are given SSA (side-side-angle, where side-angle correspond) and there are different types of solutions. You can have no possible triangle, one possible triangle, or two possible triangles.

GENERAL "INQUIRY" METHOD



HA HAB BAH

$$h > a \rightarrow \text{no } \Delta$$

$$h < a, a > b \rightarrow 1 \Delta$$

$$b > a > h \rightarrow 2 \Delta$$

no Δ

$$h > a$$

If h , height of the triangle, is greater than a , the triangle will not close and won't be possible.

1 Δ

$$h < a > b$$

If h is less than a , then we know our triangle "hac" will close.

BUT, if a is greater than b , the triangle will not close and won't be possible, so triangle "cbh" will not exist.



$$a = 12 \quad h < a > b$$

$$b = 10 \quad 9 < 12 > 10$$

$$h = 9$$

The pink triangle is possible.

The orange triangle is NOT possible.

2 Δ

$$b > a > h$$

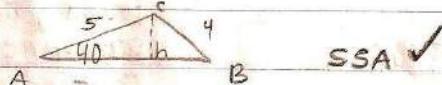
If side lengths can be ordered from greatest to least in order of b, a, h , then there are 2 triangles possible. This makes sense because if h is less than a , like in the previous solution, a triangle is possible. But, if a is also less than b , another triangle is possible. (Look at the image in the previous solution. If a was less than b AND greater than h at the same time, both pink and orange triangles would be possible.)

Cosine Law (s)

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

• $\angle A = 40^\circ$ $a = 4$ $b = 5$



SSA ✓

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 40 = \frac{h}{5} \quad 3.2 > 4 \quad \times$$

$$5 \sin 40 = h \quad 3.2 < 4 > 5 \quad \times$$

$$3.2 = h$$

$$5 < 4 < 3.2 \quad \checkmark$$

① Determine if this is an SSA scenario by drawing a picture.

② Find h to see how many triangles there are

③ Use HA HAB BAH

to determine how many triangles there are

④ Decide which inequality is correct

∴ there are 2 triangles because $b < a < h$

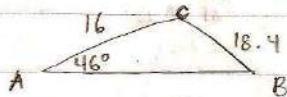
⑤ Since BAH is correct, there are 2 triangles.

Write ∴ statement.

(8b) NO Ambiguous case → using cosine law

If you use cosine law on a triangle that can be solved using sine law, you will not have to worry about ambiguity.

• $A = 46^\circ$ $b = 16$ $a = 18.4$ $C = ?$ $c = ?$ $B = ?$



① Draw your triangle

② Use cosine law to solve for side c .

(You need this if you want to solve for angle C)

NOTE: When solving for C using cosine, use a^2 and $\cos A$ because in your triangle, those are the only two that correspond.

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

$$(18.4)^2 = (16)^2 + c^2 - 2(16)(c)(\cos 46)$$

$$0 = c^2 - (2)(16)(\cos 46)c + (16)^2 - (18.4)^2$$

$$0 = c^2 - 22.2c - 82.56$$

(QUAD FORMULA)

$$c = 25.5$$

$$c = -3.25$$

③ note: your c has become an input you can FACTOR. so, bring all terms to one side.

If it has no real answers, triangle doesn't exist.

If, when you factored,

you got 2 possible solutions, there are 2 triangles.

Here, there is only one.

not a possible side length

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{(18.4)^2 + (16)^2 - (25.5)^2}{2(18.4)(16)}$$

$$\cos C = -0.0945\dots$$

$$C = \cos^{-1}(-0.0945\dots)$$

$$C = 95.4$$

$$C = 95^\circ$$

when you
did \cos^{-1}

here, don't worry
about ambiguity
because you already
eliminated/determined
that there is only
1 C -value.

④ Now that you have found your side c , use cosine law to solve for your angle C . (use re-arranged cosine law)

$$180^\circ - 95^\circ - 46^\circ = B$$

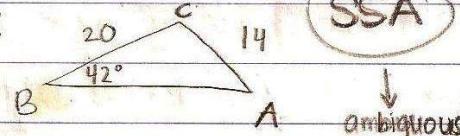
$$39^\circ = B$$

⑤ Solve for angle B using 180° property of triangles.

⑥ Ambiguous case → using sine law

$$B = 42^\circ \quad a = 20 \quad b = 14$$

$$A=? \quad C=? \quad a=?$$



SSA

↓
ambiguous
case!

① Draw your triangle

② Use sine law to find angle A since you know side a .

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin 42}{14} = \frac{\sin A}{20}$$

$$20(\sin 42) = \sin A$$

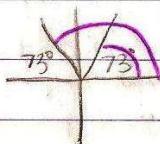
$$\sin^{-1}\left(\frac{20(\sin 42)}{14}\right) = A$$

When you do \sin^{-1} to solve for A , you have

to worry about ambiguous case.

The A that you find is actually your related acute input angle for sine (in this case)

$$73^\circ \approx A$$



③ Where is sine positive? draw 73° as related acute in these quadrants.

$$\begin{array}{l} \textcircled{1} \quad A = 73^\circ, B = 42^\circ, C = 65^\circ \\ \textcircled{2} \quad A = 107^\circ, B = 42^\circ, C = 31^\circ \end{array}$$

④ Now that you see there are two possible values for angle A, solve for each "solution set".

Because you now have 2 solution sets, you have 2 possible triangles.

This also means that since you already know sides a and b, side c will have 2 lengths. Why? Because, as shown, angle c varies depending on the quadrant location of angle A. So, to find side c, use sine law with $\sin 65$ AND THEN $\sin 31$.

⑤ Find side lengths for c in both solution sets.

using sine law:
You need to remember to have 2 possible solution sets when doing $\sin^{-1}, \cos^{-1}, \tan^{-1}$
When you have an SSA triangle!

$$\textcircled{1} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\textcircled{2} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 73}{20} = \frac{\sin 65}{c} *$$

$$c = \frac{20 \sin 65}{\sin 73}$$

$$c \approx 18.9$$

$$c = 19$$

$$\frac{\sin 107}{20} = \frac{\sin 31}{c} *$$

$$c = \frac{20 \sin 31}{\sin 107}$$

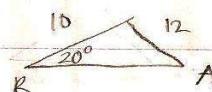
$$c \approx 10.7$$

$$c = 11$$

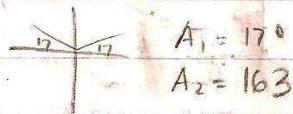
EXTRA → Using sine law to determine the number of triangles

$$\textcircled{1} \quad a = 10, b = 12, B = 20^\circ$$

① Draw triangle



② Find $\angle A$ using sine law



③ Draw on cartesian plane to find A_1 and A_2

$$\frac{\sin B}{b} = \frac{\sin A}{a} \rightarrow \frac{\sin 20}{12} = \frac{\sin A}{10}$$

$$\sin^{-1} \left(\frac{\sin 20 (10)}{12} \right) = A$$

④ Write out both sets of possible angles. Determine if possible.

$$1 \rightarrow 17^\circ + 20^\circ + 143^\circ = 180^\circ \quad \checkmark \quad \text{1 soln}$$

$$2 \rightarrow 163^\circ + 20^\circ > 180^\circ \quad \times \quad \text{NOT POSS.}$$

$17^\circ = A_1$ This is A_1 since this SSA is ambiguous case.



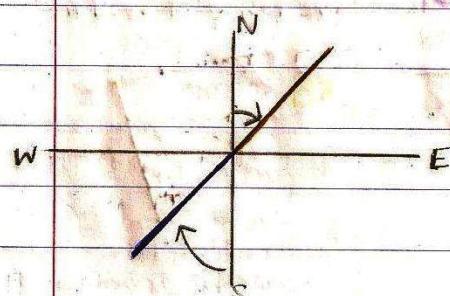
I HATE BEARINGS!

Quadrant/Compass Bearing vs. True Bearing

①(a) Types of bearing measurements

True bearing \rightarrow angle measured clockwise from North line.

Quadrant/Compass Bearing \rightarrow angle measured according to compass directions given. It is measured clockwise/counter-clockwise off of the North/South line.



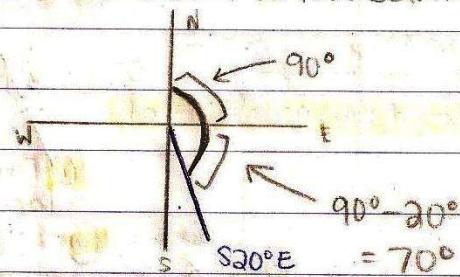
NOTE: don't confuse bearings that start from the North line with math angle rotations that start from the positive x-axis.

• 43° • $S43^\circ W$

①(b) How to convert back and forth between compass and true bearings.

• Draw $300^\circ E$ then convert to true bearing

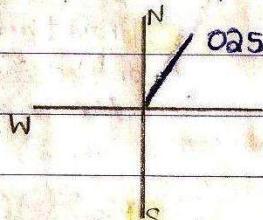
if the \Rightarrow
angle here
is two digits,
write 0 in front!
(true bearing)



① Recall that true bearing is measured clockwise from the North line. Use this knowledge + knowledge of quadrantal angles to find the true bearing.

\therefore True bearing of $300^\circ E$ is 160° .

• Draw 025° then convert to compass bearing



This bearing is:
 $\rightarrow N$
 $\rightarrow 25^\circ$ away
 $\rightarrow E$

\therefore compass bearing is $N25^\circ E$

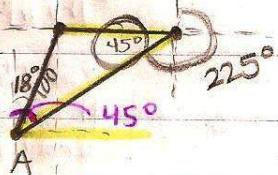
② Recall that compass bearing is measured east/west (clockwise/counterclockwise) off of North/South. Look at 025° and determine which directions to use. (use steps on left to help.)

3 steps to do this:

- \rightarrow Nor S?
- \rightarrow angle away from N/S?
- \rightarrow E or W?

(9c) Word problem using sine law + true bearings

A pilot leaves an airport and flies 100 miles at a bearing of 180° . She then detours from her plan and flies due east to drop supplies to a snowbound family. After the drop, she returns to the airport at a bearing of 225° . How far was she from the airport when she dropped supplies for the family?



① Begin by drawing an image of the question with the information given.

→ draw airport as starting point then draw crosshairs

→ draw 180° bearing that is 100 miles long and draw cross hairs

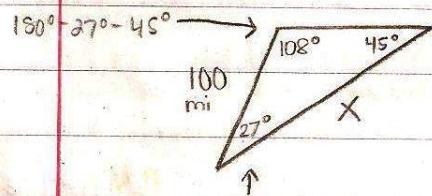
→ draw due east bearing (with unknown distance length)

then draw crosshairs

→ draw 225° bearing

② Label as many angles as you can by using angle properties.

(Z pattern) Then re-draw triangle with the unknown you need to find/calculate.



How did we get this?

By "Z" pattern, we know it was $180^\circ + x = 45^\circ$

SINE LAW! Because you have SAA.

$$\frac{\sin 45^\circ}{100} = \frac{\sin 108^\circ}{x}$$

$$\frac{x(\sin 45^\circ)}{\sin 45^\circ} = \frac{(\sin 108^\circ)100}{\sin 108^\circ}$$

$$x = \frac{(\sin 108^\circ)100}{\sin 45^\circ}$$

$$x \approx 134.49$$

$$x \approx 134.5$$

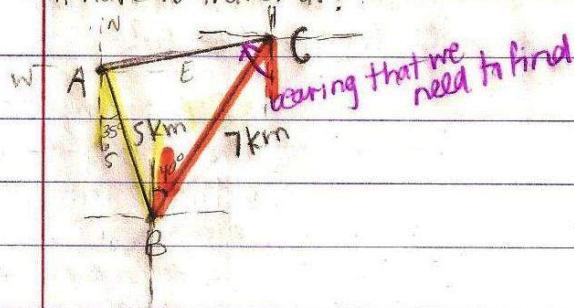
③ Decide what law to use based on information you have. Then solve for side x.

④ Write ∴ statement

∴ the pilot was 134.5 mi away from the airport when she dropped supplies for the family?

(Qd) Word problem using cosine law and compass bearings

- A ship travels S 35° E for 5km and then turns and goes N 40° E for 7km. How far would it be if the ship turned around from its point and travelled back to where it started from? At what compass bearing would it have to travel at?



① Draw an image of the information given.

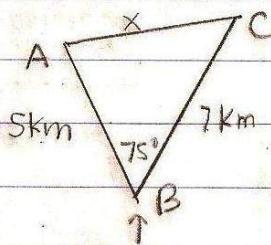
→ starting point, A then draw S 35° E and label it 5km long.

→ point B and draw crosshairs then draw N 40° E and label it 7km long

→ point C then draw crosshairs

② Using any angle properties, label as many angles as you can.

(Z pattern) Then, re-draw triangle with the unknown distance you need to find.



Using Z pattern, we found the missing piece of the entire angle B. (35° + 40°)

COSINE LAW! Because you have SAS

$$x^2 = 7^2 + 5^2 - 2(7)(5)\cos 75^\circ$$

$$x^2 = 74 - 70\cos 75^\circ$$

$$x = \pm \sqrt{74 - 70\cos 75^\circ}$$

$$x = \pm 7.47$$

$$x = \pm 7.5$$

¹
reject negative answer

∴ $x = 7.5$ km

③ Decide what law to use based on information you have. Then solve for x.

TO DO THIS: use sine law using the x you just found.

$$\frac{\sin 75}{7.47\dots} = \frac{\sin C}{5}$$

original was used →

$$\frac{5(\sin 75)}{7.47\dots} = \sin C$$

$$\sin^{-1}\left(\frac{5(\sin 75)}{7.47\dots}\right) = C$$

$$40.20^\circ = C$$

$$40^\circ = C$$

Using **Z** pattern, we know the bearing will be **$400 + 40^\circ$**

$$\therefore S80^\circ W$$

∴ It would be 7.5km if the ship travelled from where it is back to where it started from.
It would have to travel a compass bearing of **$S80^\circ W$** .

(4) Find $\angle C$ to help find the compass bearing.

(5) Using $\angle C$ and looking at the original image drawn, think about what the compass bearing from point C to point A would be.

(6) Write final **∴** statement

EXTRA: Finding θ , complicated examples

• $2\sin^2 \theta - 1 = 0$

$\sin^2 \theta = \frac{1}{2} = \frac{y}{r}$

$\sin \theta = \pm \frac{1}{\sqrt{2}}$



Because $\pm \frac{1}{\sqrt{2}}$, there are 4 θ possibilities.

Sine can be positive in I and II and sine can be negative in III and IV.

$\theta = 45^\circ$ or 135° or 225° or 315° .

• $\tan(\theta + 20^\circ) = -3.65$

let $\theta + 20^\circ = a$

$\tan a = -3.65$

$a_r = \tan^{-1}(-3.65)$

$a_r \approx 75^\circ$

θ

75°

θ

75°

$a_1 = 105^\circ$

$a_2 = 285^\circ$

$a_3 = 105^\circ$

$a_4 = 285^\circ$

► to find θ :

Sub $a + 20^\circ$ back into a_1 and a_2 .

$a_1 = 105^\circ$

$a_1 + 20^\circ = 125^\circ$

$\theta = 85^\circ$

$a_2 = 285^\circ$

$a_2 + 20^\circ = 305^\circ$

$\theta = 265^\circ$