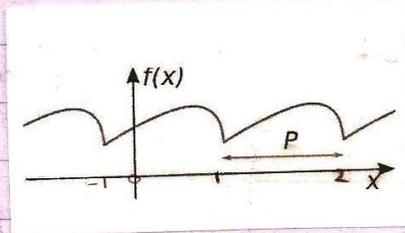


Intro to sinusoidal graphs

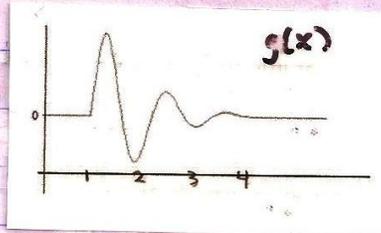
1.1 Definitions

- Periodic function** → a function whose graph repeats at regular intervals. (Intervals are called cycles)
The max/min points of cycles will be the same if the function is periodic.

Also: Aperiodic function has y-values that show a repetitive pattern ONLY when the x-values are constant.

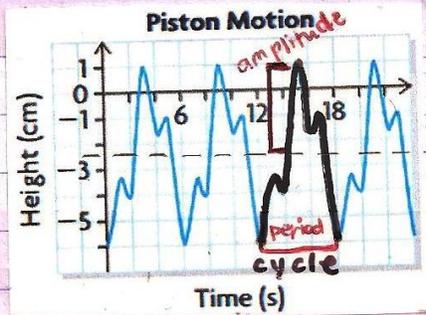


This function $f(x)$ is periodic because its cycle, P , repeats at regular intervals. And, all max/min points are the same.



This function, $g(x)$ is not periodic because it doesn't repeat at regular intervals.

This is another example of a periodic function →



Period → the change in the independent variable that corresponds to one cycle of the function.

$$P = (\text{last horiz. pt of cycle}) - (\text{first horiz. pt of cycle})$$

For this graph:
 $P = (18) - (12) = 6$
 $P = 6$

★ PERIOD IS ALWAYS POSITIVE!

think "average" → **Axis** → equation of the horizontal line half way between the maximum point and minimum point.

$$C = \frac{(\text{Max}) + (\text{Min})}{2}$$

For this graph:

$$C = \frac{(1) + (-6)}{2} = \frac{-5}{2} = -2.5$$

$$C = -2.5$$

Amplitude formulas

$$a = (\text{Max}) - (C)$$

$$a = (C) - (\text{Min})$$

$$a = \frac{(\text{Max}) - (\text{Min})}{2}$$

Amplitude → Half the distance between the maximum point and minimum point.

ALSO: The distance between the axis to the maximum/minimum point.

For the graph on previous page:

$$a = (\text{Max}) - (c)$$

$$a = (1) - (-2.5)$$

↑
keep negative!

$$a = -3.5$$

$$a = (c) - (\text{Min})$$

$$a = (-2.5) - (-6)$$

$$a = 3.5$$

$$a = \frac{(\text{Max}) - (\text{Min})}{2}$$

$$a = \frac{(1) - (-6)}{2} = \frac{7}{2} = 3.5$$

$$a = 3.5$$

★ AMPLITUDE IS ALWAYS POSITIVE

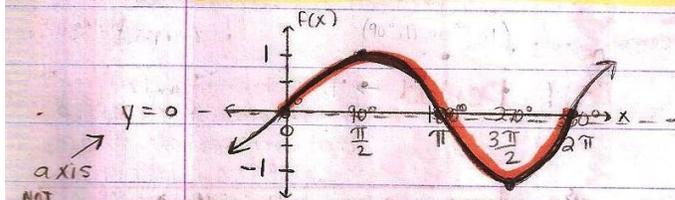
Range $\{ \text{Min} \leq y \leq \text{Max} \}$

For graph on previous page:

$$\{ -6 \leq y \leq 1 \}$$

Sinusoidal → in the shape of/resembles a sine/cosine graph because it is wavelike in structure. Only sine and cosine graphs are wavelike (trigonometric) functions.

(1b) Cycle of the parent sine function



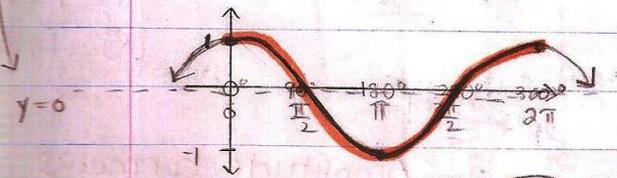
$$f(x) = \sin(x)$$

↑ ratio of trig definition on the UNIT CIRCLE
↑ angle

SINE STARTS AT THE AXIS THEN GOES UP.

Period: $p = 2\pi$ Range: $[-1, 1]$
Axis: $c = 0$
Amplitude: $a = 1$
Symmetry: Odd

Cycle of the parent cosine function



$$f(x) = \cos(x)$$

↑ ratio of trig definition on the UNIT CIRCLE
↑ angle

COSINE STARTS AT MAX THEN GOES DOWN.

Period: $p = 2\pi$ Range: $[-1, 1]$
Axis: $c = 0$
Amplitude: $a = 1$
Symmetry: Even

$$y = a f(k(x+d)) + c$$

②a Transformations of sine/cosine graphs

• $y = -2 \cos(0.75x - 45)^\circ + 3$

$$y = -2 \cos[0.75(x-60)]^\circ + 3$$

transformations of parent $y = \cos x$

$a = -2$ (stretch, vertical, reflect in x -axis)

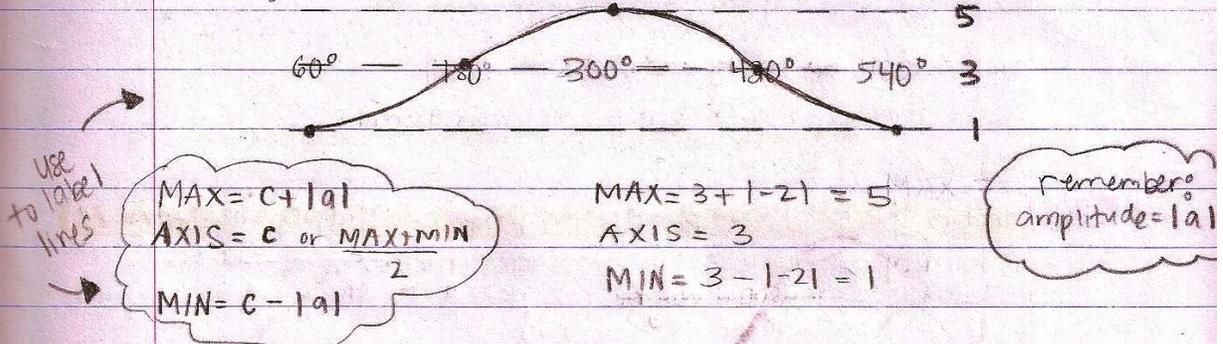
$k = 0.75$ (stretch, horizontal)

$d = 60^\circ$ (shift right)

$c = 3$ (shift up)

① state all transformations (may need to factor out k -value to see d -value)

② Sketch 3 horizontal dotted lines and label values using Max, min, and axis. (use formulas to find max/min because axis is given with the c -value. Use amplitude to find max/min)



③ Decide on the period of the transformed graph by using:

$$P = \frac{360^\circ}{|k|} \text{ or } \frac{2\pi}{|k|}$$

← This formula for period is true for sine, cosine, cosecant, secant.

$$P = \frac{360^\circ}{0.75} = 480^\circ \quad \text{period} = 480^\circ$$

④ Use your new period to determine where 1 cycle of the graph starts and ends. (Apply d -value here)

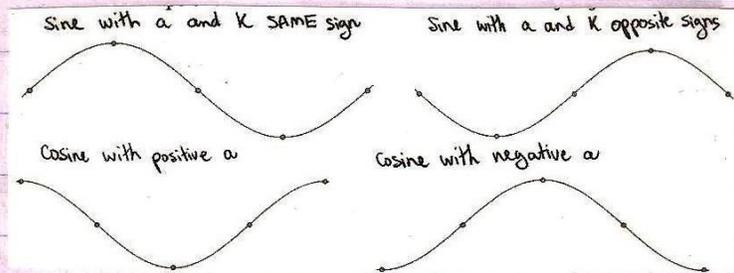
FIRST POINT OF CYCLE → $d = 60^\circ$

LAST POINT OF CYCLE → $d + \text{period} = 60^\circ + 480^\circ = 540^\circ$

Label these two points on the axis

⑤ Find the quarters of the cycle by finding the middle of the period then the middle of each 'half'. (OR, count by $\frac{1}{4}$ period)

- 6) Decide on the shape of the wave that you have to use for your transformed sinusoidal. After you have chosen the shape, follow the pattern of where the points are plotted in relation to their quarters. Points are ONLY either on the MAX, AXIS, or MIN.



sine always \rightarrow axis to axis

cosine always \rightarrow max to max or min to min

- 2b) What is the relationship between period and frequency? ("k-value")

FREQUENCY \rightarrow k-value tells you how long a period in a transformed sinusoidal will be compared to the parent graph's period of 360° (or 2π)

$P = \frac{360^\circ}{|k|}$ or $\frac{2\pi}{|k|}$ adjusts the period of the transformed sinusoidal by dividing the original period by the frequency, k. (absolute value of k because period is never negative!)

* period + frequency are reciprocals of each other.

• $P = \frac{360^\circ}{|k|}$

• find k if period = 120°

Find period if $k = -3$

$P = \frac{360^\circ}{|k|} \rightarrow k = \frac{360^\circ}{P}$

$P = \frac{360^\circ}{|-3|} = 120^\circ$

$k = \frac{360^\circ}{120^\circ} = 3$

$P = 120^\circ$

$k = 3$

GENERAL STEPS FOR SKETCHING TRANSFORMED SINUSOIDALS

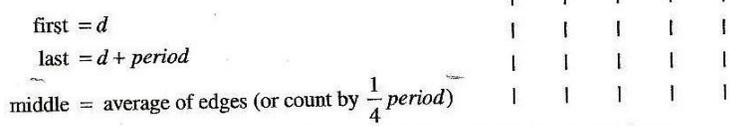
- Sketch 3 horizontal dotted lines and label the values on them (Don't bother drawing the x and y axes)

$$\begin{aligned} \text{MAX} &= c + |a| \text{ -----} \\ \text{axis} &= c = \frac{\text{MAX} + \text{MIN}}{2} \text{ -----} \\ \text{MIN} &= c - |a| \text{ -----} \end{aligned}$$

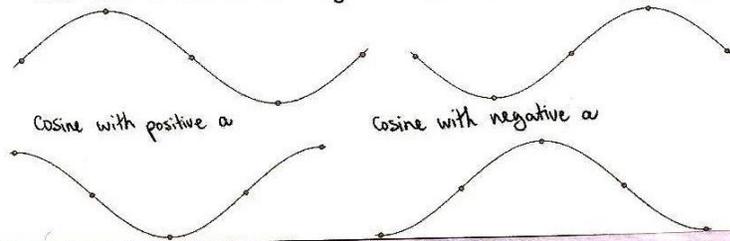
recall that

$$\begin{aligned} \text{amplitude} &= |a| \\ \text{period} &= p = \frac{360^\circ}{|k|} = \frac{2\pi}{|k|} \end{aligned}$$

- Sketch in (to erase later, not VAs) the 5 vertical lines to clearly see quarters of the cycle and label them as follows



- Decide on the shape of the wave and sketch it between the lines you just drew



Sinusoidal Word Problems

(4a) How to find the period and the k-value given revolutions per second

- 960 revolutions per second

This ratio is known as the frequency.

THINK: frequency is measured in $\frac{1}{\text{sec}}$ and the period would be measured in sec. \therefore reciprocals!

As noted in journal # (2b), frequency and period are reciprocals of each other.

$$f = 960$$

$$P = \frac{1}{f} \quad P = \frac{1}{960}$$

① Using relationship between period and frequency, find the period.

② Using the relationship between k-value and period, find k-value.

sine/cosine/second/cosecant

$$k = \frac{2\pi}{P} \text{ or } \frac{360^\circ}{P}$$

$$k = \frac{360^\circ}{\left(\frac{1}{960}\right)} = 345600$$

$$\therefore k = 345600^\circ$$

tangent/cotangent

$$k = \frac{\pi}{P} \text{ or } \frac{180^\circ}{P}$$

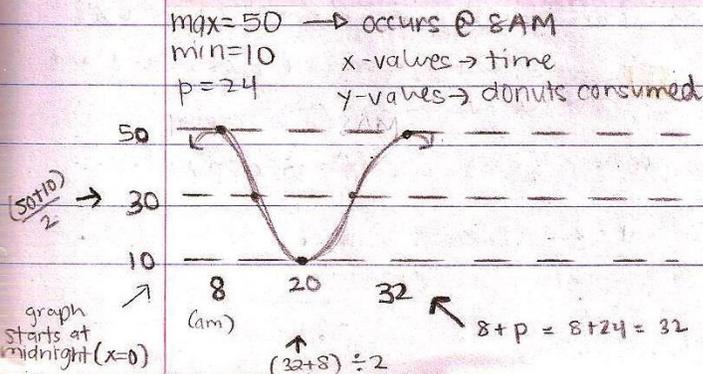
$$k = \frac{180^\circ}{\left(\frac{1}{960}\right)} = 172800$$

$$\therefore k = 172800$$

(4b) How to sketch a wave given description in a word problem + how to find its equation

Example 3. You consider getting yourself a baby goat of your own. After your parents insist that you pay for the goat's food, you get a job as manager of Krispy Kreme. As part of your responsibility, you have to find a sinusoidal model for the number of donuts consumed throughout the day (and night, since Krispy Kreme is open 24 hours). After an intense data collection session, you find that the maximum number of donuts is 50, the minimum is 10, the patterns repeat every 24 hours, and that the peak consumption occurs at 8 AM.

- Sketch a graph that represents the donut consumption with respect to the number of hours since Sunday at midnight.
- Find the equation of the sinusoid.



① Write down what you know from the question.

② Draw graph using this information.

*Since max (50) occurs at 8 AM, use this as your first point.

→ Use cosine because of the shape of the wave.

$$a = 20$$

$$c = 30$$

$$d = -8$$

$$k = \frac{360^\circ}{P} = \frac{360^\circ}{24} = 15^\circ$$

$$k = 15^\circ$$

$$\therefore y = 20 \cos(15^\circ(x-8)) + 30$$

③ To find equation, determine if you will use sine/cosine and begin to look for variables a, k, d, c

④ Combine variables into one equation

④c How to find y if given x (continuation of word problem in ④b)

c. You want to know how many people will be having donuts for breakfast (not a very nutritional meal!) at 5 AM. Find the exact value of $f(5)$.

$$y = 20 \cos(15^\circ(x-8)) + 30$$

$$y = 20 \cos(15^\circ(5-8)) + 30$$

$$y = 20 \cos(15^\circ(-3)) + 30$$

$$y = 20 \cos(-45^\circ) + 30$$

this is special and can be solved using exact values

$$\cos -45^\circ = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = 20 \left(\frac{\sqrt{2}}{2} \right) + 30$$

$$y = 10\sqrt{2} + 30$$

① They tell you $x=5$, so plug in 5 for x into your equation to find the # of donuts consumed at 5 AM.

($x=5$ because 5 AM is 5 hours more than midnight, where the graph starts)

Then, solve for y .

② Use exact values!

④d How to find ALL x 's if given y from the equation (continuation of word problem in ④b)

d. You will need extra employees when your donut consumption level is greater than or equal to 40. Find a general solution to the equation $f(x) = 40$.

① Here we are given a y -value and we need to find all the points on the graph where $y=40$. To solve for x ONLY in our equation, we have to do a replacement. Let $\theta = 15^\circ(x-8)$. Also, make the equation $y=40$ so we can find those specific x 's.

$$40 = 20 \cos \theta + 30$$

$$\frac{10}{20} = \cos \theta$$

$$\frac{x}{r} = \frac{1}{2} = \cos \theta$$

special

$$\theta = 60^\circ$$

$$\theta_1 = 60^\circ$$

$$\theta_2 = -60^\circ$$

② Using θ_1 and θ_2 , replace back in for θ and solve for x .

$$\frac{60^\circ}{15^\circ} = \frac{15^\circ(x-8)}{15^\circ}$$

$$4 = x - 8$$

$$12 = x$$

$$\pm 24$$

$$\frac{-60^\circ}{15^\circ} = \frac{15^\circ(x-8)}{15^\circ}$$

$$-4 = x - 8$$

$$4 = x$$

$$\pm 24$$

when $y = 40$

$$x_n = 12 + 24n \quad \text{or} \quad x_n = 4 + 24n \quad \{n \in \mathbb{N}\}$$

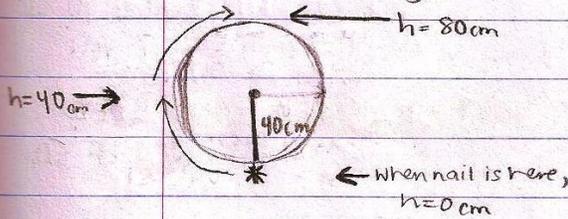
The general solution tells all the times (after Sunday at midnight) where donut consumption = 40.

③ Realize that when $x = 4$ and $x = 12$, donut consumption is equal to 40.

To find general solution: take each x value then create an arithmetic sequence where d is the period of 24 hours.

EXTRA: WORD PROBLEM TO MODEL USING SINUSOIDALS

- Car is travelling at 60km/h and a nail gets stuck in the tire. The tire has a radius of 40cm.
 - Sketch a graph of the height versus distance travelled by the nail for one cycle (in cm).
 - Sketch a graph of height versus time travelled by the nail for one cycle. (in sec)
 - create an equation for your graph in a).

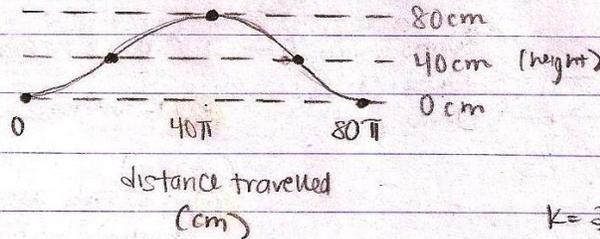


① Part a) only asks for height/ distance graph. To determine height(s), draw a diagram of the tire.

② Find the period of one cycle. Think: in terms of this question, what would one cycle of this graph be? One cycle = one full rotation of this tire. A full rotation's distance is aka the CIRCUMFERENCE of the circle.

Find circumference! $C = 2\pi r = 2\pi 40 = 80\pi$

③ Sketch a graph and label points you know. Then connect with a sinusoidal wave.



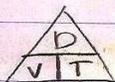
④ List variables of an equation you know + find any if needed.

$a = 40$	$c = 40$	$d = 0$
----------	----------	---------

* reflected cosine

$$k = \frac{2\pi}{P} = \frac{2\pi}{80\pi} = \frac{1}{40}$$

$k = \frac{1}{40}$



$$T = \frac{D}{V}$$

← distance
← speed

We know $D = 80\pi \text{ cm}$ & $V = 60 \text{ km/h}$
 But our units for D and V are very different. So, before using $T = \frac{D}{V}$, let's convert D and V into cm and sec .

$80\pi \text{ cm}$ DON'T NEED TO CHANGE

$$1. \quad \frac{60 \text{ km}}{\text{h}} \times \frac{1 \text{ h}}{60 \text{ min}} = \frac{60 \text{ km}}{60 \text{ min}}$$

↑
 put hours on top to cancel.

$$2. \quad \frac{60 \text{ km}}{60 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ sec}} = \frac{60 \text{ km}}{3600 \text{ sec}}$$

$$3. \quad \frac{60 \text{ km}}{3600 \text{ sec}} \times \frac{1000 \text{ m}}{1 \text{ km}} = \frac{60000 \text{ m}}{3600 \text{ sec}}$$

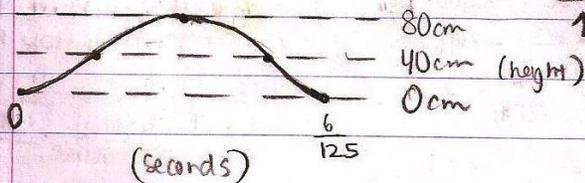
↑
 put km on bottom to cancel

$$\frac{60000 \text{ m}}{3600 \text{ sec}} \times \frac{100 \text{ cm}}{1 \text{ m}} = \frac{6000000 \text{ cm}}{3600 \text{ sec}} \Rightarrow \boxed{\frac{5000 \text{ cm}}{3 \text{ sec}}}$$

This is our speed converted to units that we want to use.

$$T = \frac{D}{V} = \frac{80\pi \text{ cm}}{\left(\frac{5000 \text{ cm}}{3 \text{ sec}}\right)} = \frac{80\pi \text{ cm} \cdot 3 \text{ sec}}{5000 \text{ cm}} = \frac{240 \text{ sec}}{5000} = \boxed{\frac{6}{125}}$$

⑦ Find period in t seconds using $T = \frac{D}{V}$



THIS IS 1 PERIOD in t -seconds

⑧ Re-sketch graph

$$\therefore y = 40 \cos \frac{1}{40}x + 40 \quad (\text{for part A})$$

⑨ Create an equation for part a) using step 4.

⑤ For part b), we are being asked to convert the unit of our x -axis from distance to time.

How do we convert distance to time?

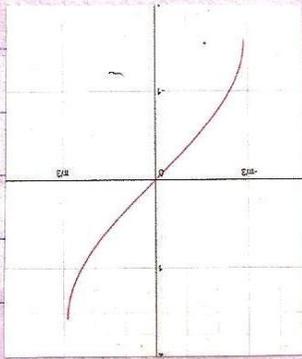
② To convert, do step-by-step conversions to only denominator/numerator. (One at a time)

★ ALWAYS MULTIPLY BY A RATIO EQUAL TO 1 WHERE THE UNIT YOU WANT TO CONVERT IS BEING CANCELLED AND THUS, CONVERTED.

REMEMBER:
 $\sin(\text{angle}) = \text{ratio}$
 $\sin^{-1}(\text{ratio}) = \text{angle}$

Inverse Trig Functions

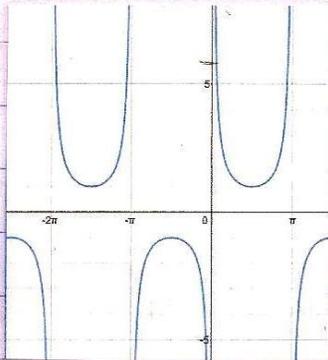
⑤ a) Why is $\arcsin x$ a better notation for inverse sine than $\sin^{-1}x$?



$$y = \sin^{-1}(x)$$



$$y = \arcsin(x)$$

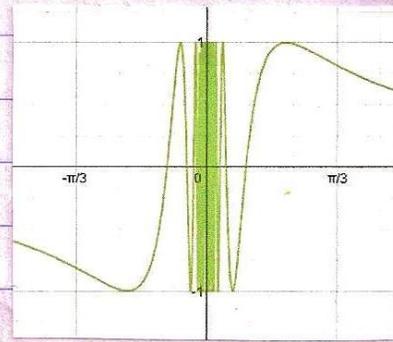


$$y = (\sin x)^{-1}$$



(cosecant)

$$y = \csc(x)$$



$$y = \sin(x^{-1})$$

∴ $\arcsin(x)$ is a better notation because the -1 power can easily be mistaken to be on the entire function of $\sin(x)$ or can be confused to just be on the (x) .

⑤ b) When do primary trig and inverse trig "cancel" each other?

$$\bullet \sin^{-1}\left(\sin \frac{\pi}{10}\right) = \frac{\pi}{10}$$

$$\text{LS} = \sin^{-1}\left(\sin \frac{\pi}{10}\right)$$

$$= \sin^{-1}\left(\frac{y}{r}\right) = \frac{\pi}{10}$$

THINK: WHY DOES THIS WORK?

$\sin \frac{\pi}{10}$, our input, is a y-value/ratio that is in quadrant I of the unit circle. When \sin^{-1} is done on a ratio in this quadrant, the same angle used to obtain our ratio will be our final output.

This statement is true because sine and arcsin cancel each other. To prove, let's evaluate the LS using process of simplifying.

① Evaluate $\sin \frac{\pi}{10}$ using general variables.

② Recognize that $\sin^{-1}(y)$ is looking for an angle output that has a ratio of y . Based on previous step, we know that $\frac{\pi}{10}$ will be θ .

• $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}$

LS = $\sin^{-1}\left(\frac{1}{2}\right)$

$\sin\theta = \frac{1}{2} = \frac{y}{r}$

This is a special Δ case and we know how to solve for θ .

$\therefore \theta = \frac{\pi}{6}$ $\star \theta$ is only $\frac{\pi}{6}$ because \sin^{-1} only gives output angles in quadrant I or IV.

This statement is true because in this case, sine and arcsine DON'T cancel each other. To prove, let's evaluate LS using process of simplifying.

① Evaluate $\sin\frac{5\pi}{6}$

② Rewrite \sin^{-1} in a way that you know what to be looking for.

THINK: WHY DOES THIS HAPPEN?

$\sin\frac{5\pi}{6}$, our input, was a ratio located in quadrant II. \sin^{-1} only has output angles located in quadrant I and III. So, it took the ratio that was in quadrant II and based on the ratio's related acute, gave an angle equal to $\frac{5\pi}{6}$ (based on θ_r) that was in quadrant I.

• $\cos(\cos^{-1}(0.2)) = 0.2$

THINK: WHY DOES THIS WORK?

Ratio of 0.2 is being used as an input to find its corresponding θ .

The θ (output) that is found becomes the input for cosine. Cosine of this angle θ will produce an output ratio of this angle that we saw in the previous step was 0.2. This is why \cos and \cos^{-1} "cancel."

① In words, think about what is happening on the LS to make the RS equal to 0.2

• $\cos(\cos^{-1}(10)) = \text{undefined}$

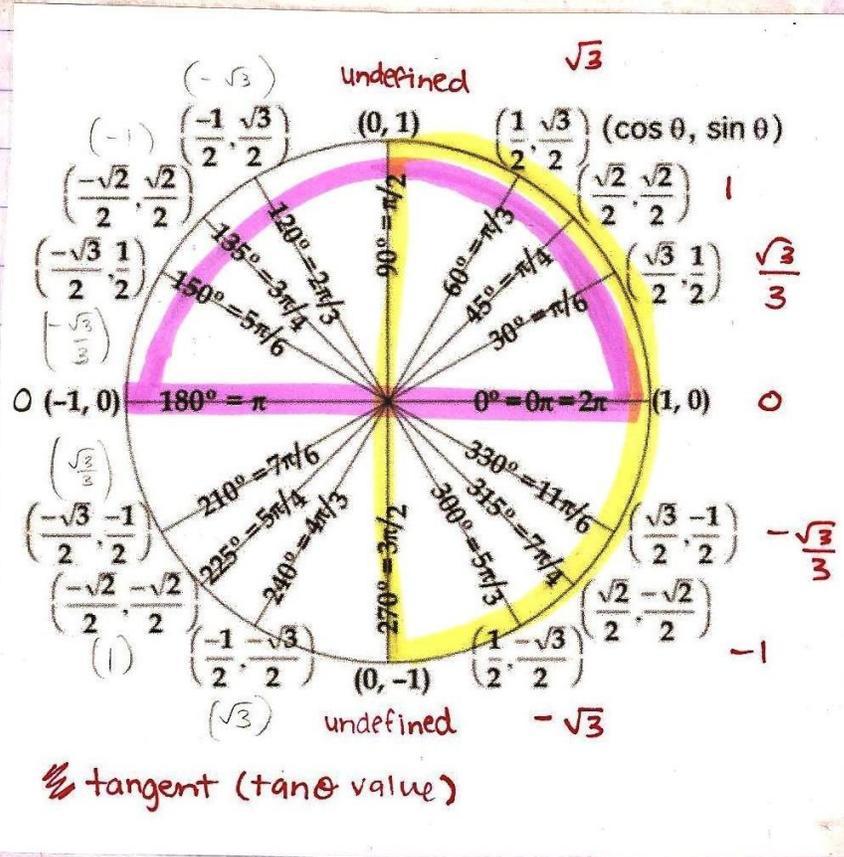
↑
The input ratio of \cos^{-1} is 10, so it doesn't satisfy the domain of this function.

\therefore undefined

① This is undefined because the domain of arccosine goes from -1 to 1.

(5e) Unit Circle with points based on $\cos \theta$ and $\sin \theta$ of the special angles

arcsine
arctangent
arccosine



NOTE:
numbers in red
are values
of the (x, y)
coordinates
that have
already
been divided
because:
 $\tan = \frac{y}{x}$

TO USE THIS UNIT CIRCLE:

• $\text{Arctan } \sqrt{3} = \frac{\pi}{3} = 60^\circ$

- ① Look for a ratio that is $\sqrt{3}$ in quad I, IV
- ② Corresponding angle is your answer

• $\text{Arccos } \frac{\sqrt{2}}{2} = \frac{\pi}{4} = 45^\circ$

- ① Look for an x-value that is $\frac{\sqrt{2}}{2}$ in quad I, II
- ② The corresponding angle is your answer

• $\text{Arcsin } -\frac{1}{2} = \frac{11\pi}{6} = 330^\circ$

- ① Look for a y-value that is $-\frac{1}{2}$ in quad I, IV
- ② Corresponding angle is your answer