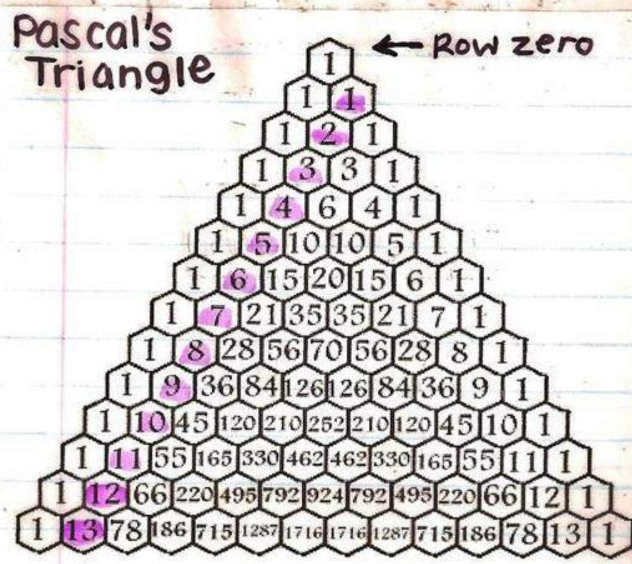
① a) Using Pascal's Triangle to expand binomials

• $(1-3x)^6$



① Look at the exponent on the binomial. This tells you the row you need to look at in Pascal's Triangle.

Row 6 : 1/6/15/20/15/6/1

② Set up empty brackets with each coefficient from row 6.

$1(\ )(\ ) + 6(\ )(\ ) + 15(\ )(\ ) +$
$20(\ )(\ ) + 15(\ )(\ ) + 6(\ )(\ ) +$
$1(\ )(\ )$

③ Write powers on each bracket. The first bracket of every pair will start with the power of the binomial and decrease to zero. The second bracket of each pair will do the OPPOSITE.

(note: degree of each term = exponent on binomial)

$$1(\ )^6(\ )^0 + 6(\ )^5(\ )^1 + 15(\ )^4(\ )^2 + 20(\ )^3(\ )^3 + 15(\ )^2(\ )^4 + 6(\ )^1(\ )^5 + 1(\ )^0(\ )^6$$

④ Now, in each bracket, fill in the terms. Every first bracket will have the first term. Every second bracket in each term will have the second term.
☆REMEMBER TO KEEP ⊖ with THE 3X. And then, simplify.

$$1(1)^6(-3x)^0 + 6(1)^5(-3x)^1 + 15(1)^4(-3x)^2 + 20(1)^3(-3x)^3 + 15(1)^2(-3x)^4 + 6(1)^1(-3x)^5 + 1(1)^0(-3x)^6$$

$$= 1 - 18x + 135x^2 - 540x^3 + 1215x^4 - 1458x^5 + 729x^6$$

$(4+\frac{2}{x})^3$

① The exponent on the binomial tells us to look at row 3 in Pascal's Triangle.

Row 3: 1/3/3/1

② Now, set up brackets with each of the coefficients. Also, write the powers on all brackets. Remember: 1st bracket of each pair begins with power 3 and decreases to 0. 2nd bracket of each pair begins with 0 and increases to 3.

$1(\ )(\ ) + 3(\ )(\ ) + 3(\ )(\ ) + 1(\ )(\ )$

$= 1(4)^3\cancel{(\frac{2}{x})^0} + 3(4)^2(\frac{2}{x})^1 + 3(4)^1(\frac{2}{x})^2 + 1\cancel{(4)^0}(\frac{2}{x})^3$

$= 64 + 48(\frac{2}{x}) + 12(\frac{4}{x^2}) + (\frac{8}{x^3})$

$= 64 + \frac{96}{x} + \frac{48}{x^2} + \frac{8}{x^3}$

(1b) How to use differences to find the equation of a polynomial (when given a table of values)

x	1	2	3	4	5	6
y	-5	0	1	-8	-33	-80

Δy

+5, +1, −9, −25, −47

$\Delta\Delta y$

−4, −10, −16, −22

$\Delta\Delta\Delta y$

−6, −6, −6

✓ cubic

① Find 1st differences. If they are not constant, function is not linear.

② Find 2nd differences. If they are not constant, function is not quadratic.

③ Find 3rd differences. If they are constant, function is cubic.

④ You need to find the equation of this function in $y=ax^3+bx^2+cx+d$ form. To do this, set up a special table. (use 4 x-values in this case because it is equal to number of terms)

⑤ In the y column, plug in each x-value

⑥ In the Δy column, find the change in the expressions of y.

⑦ Do the same for columns $\Delta\Delta y$ and $\Delta\Delta\Delta y$

$y=ax^3+bx^2+cx+d$

x	y	Δy	$\Delta\Delta y$	$\Delta\Delta\Delta y$
1	a+b+c+d			
		7a+3b+c		
2	8a+4b+2c+d		12a+2b	
		19a+5b+c		6a
3	27a+9b+3c+d		18a+2b	
		37a+7b+c		
4	64a+16b+4c+d			
		1st diff	2nd diff	3rd diff

$\Delta\Delta\Delta y = \Delta\Delta\Delta y$

$\frac{6a}{6} = \frac{-6}{6}$

$a = -1$

⑧ By looking at column $\Delta\Delta\Delta y$, we see that only $6a$ is left. So, $6a$ must be equal to the 3rd differences constant. Make $6a$ equal to -6 and solve for a

*you Know $a=-1$ so, sub in!

$\Delta\Delta y = \Delta\Delta y$

$12a + 2b = -4$

$12(-1) + 2b = -4$

$-12 + 2b = -4$

$\frac{2b}{2} = \frac{8}{2}$

$b = 4$

⑨ By looking at column $\Delta\Delta y$, we see that ~~there are 2 expressions. The first~~ expression corresponds with the first number in the 2nd differences. Make $12a + 2b$ equal to -4 and solve for b

$\Delta y = \Delta y$

*you Know $a=-1$ $b=4$ so, sub in!!

$7a + 3b + c = 5$

$7(-1) + 3(4) + c = 5$

$-7 + 12 + c = 5$

$c = 0$

⑩ By looking at column Δy, we see that there are 3 expressions. The first ~~expression corresponds with the~~ first number in the 1st differences. Make $7a + 3b + c$ equal to 5 and solve for c.

$y = ax^3 + bx^2 + cx + d$

→ if $x = 1$

$y = a + b + c + d$

→ when $x = 1$, $y = -4$. and we Know $a = -1$ $b = 4$ $c = 0$

$-4 = -1 + 4 + 0 + d$

$-7 = d$

⑪ By looking at column y, we are given expressions that represent the y-values of each corresponding x-value. Use any expression to solve for d by sub-ing in $a=-1$, $b=4$, $c=0$ (which you already found) You Know what y is for each expression so you can sub it in.

∴ the equation of this function is

$$y = -x^3 + 4x^2 - 7$$

↳ original: $y = -x^3 + 4x^2 + 0x - 7$ but, we don't write "0x"

2a) Patterns → some common vocab

Arithmetic - the adding/subtracting of the same value to each term to determine the next term.

In an arithmetic pattern, there is a common difference that can be found by subtracting a term by its previous term.

↳ $d = a_n - a_{n-1}$

$\overset{-6}{\frown}\ \overset{-6}{\frown}\ \overset{-6}{\frown}\ \overset{-6}{\frown}\ \overset{-6}{\frown}$

- 37, 31, 25, 19, 13, 7

① See if there is a common difference between each term.

$d = a_n - a_{n-1}$

$d = 31 - 37 = -6$ ∴ the common difference $= -6$

This sequence is arithmetic because you need to subtract 6 each time to get the next term.

Geometric – the multiplying/dividing of the same value to each term to determine the next term.

In a geometric pattern, there is a common ratio that can be found by dividing a term by its previous term.

$\rightarrow r = \frac{a_n}{a_{n-1}}$

- $9, 3, 1, \frac{1}{3}, \frac{1}{9}$

$\div 3 \quad \div 3 \quad \div 3 \quad \div 3$

① See if there is a common ratio between each term.

$r = \frac{a_n}{a_{n-1}}$ ∴ common ratio is $\frac{1}{3}$.

$r = \frac{3}{9} = \frac{1}{3}$

This sequence is geometric because you need to divide by 3 each time to get the next term. (OR → because the ratio of each term and its previous term is $\frac{1}{3}$.)

Finite – a pattern that has a specific number of terms and doesn't continue forever.

- 1, 3, 5, 7, 9 ← This sequence only has 5 terms then stops

Infinite – a pattern that continues forever.

- 1, 3, 5, 7, 9... ← Now, because this sequence is followed by (...), it is infinite and never stops.

Sequence – a list of numbers linked together through a pattern. (aka an ordered list of terms)

- 2, 4, 8, 16, ...

2, 4, 6, 8

1, 1, 2, 3, 5, ...

$a_1, a_2, a_3, \ldots$

These are all sequences

Each term in a sequence is identified by its location in the sequence.

a_n ← the subscript n tells us the location of term a in a sequence

- 1, 3, 5, 7... a_3 = 3rd term = 5

*NOTE: DOMAIN/RANGE OF A SEQUENCE
D: the # of terms (subscript values)
R: the terms (term values)

Series - The addition of numbers (aka the addition of a sequence of numbers)

- 1+3+5
  1+2+3+4
  $\sum_{n=3}^{5} n$ ← sigma will be explained later in #2c.

  These are series

SERIES ≠ SEQUENCES

Recursive Formula - Each term in a sequence is based on its previous term.

Because the next term of these sequences are based on the previous term, in order to find ANY term, you need the term that comes right before it.

HINTS: a formula is recursive if the subscript has more than 1 term.
also, a formula is recursive if you are given the first/few terms of a sequence.

- $a_n = 4a_{n-1} - 3$, $a_1 = 3$ ← recursive because subscript n-1 appears and because first term is given.

Explicit Formula - Each term in a sequence is based on its location.

A formula is explicit if the subscript only has 1 term (variable)
Using an explicit formula, we can directly find any term in a sequence.

- $a_n = 3n - 5$

$a_3 = 3(3) - 5$
$= 9 - 5$
$= 4$

∴ a_3 (the third term of the sequence) is equal to 4.

① If you want the 3rd term, n=3. So, it would be a_3 (a_3 = location of term = 3rd position)

2b) How to use a recursive formula to find terms of a sequence

- $a_n = a_{n-2} + a_{n-1}$, $a_1 = 1$, $a_2 = 2$
  Find the first 5 terms of this sequence.

a_3	a_4	a_5
$a_3 = a_{3-2} + a_{3-1}$	$a_4 = a_{4-2} + a_{4-1}$	$a_5 = a_{5-2} + a_{5-1}$
$a_3 = a_1 + a_2$	$a_4 = a_2 + a_3$	$a_5 = a_3 + a_4$
$a_3 = 1 + 2$	$a_4 = 2 + 3$	$a_5 = 3 + 5$
$\boxed{a_3 = 3}$	$\boxed{a_4 = 5}$	$\boxed{a_5 = 8}$

① Recognize that they have already given you the first two terms, a_1 and a_2.
② Find a_3. When finding a_3, n=3. In order to find a_3, you NEED a_1 and a_2. Since they are given, you can solve.
③ Find a_4 and a_5 by following steps previously used to find a_3.

*SIGMA can distribute across addition/subtraction. NOT multiplication/division.

$$\sum_{n=2}^{10}(n+5) = \sum_{n=2}^{10} n + \sum_{n=2}^{10} 5$$

*SIGMA NOTATION IS USED FOR SERIES!

2c) Sigma Notation — what does it mean?

$$\sum_{n=\square}^{x} n$$

This notation says that you must evaluate $n=\square$ until the value of x. You must evaluate these numbers in the function/equation provided.

After you evaluate all the values, you must ADD all your answers.

HOW TO USE IT:

- $$\sum_{n=2}^{10}(n+5)$$

$= (2+5)+(3+5)+(4+5)+(5+5)+(6+5)+(7+5)+(8+5)+(9+5)+(10+5)$

$= 7+8+9+10+11+12+13+14+15$

$= 99$

this is a SERIES

1. Recognize what values of n you need to start/end with. $n=2 \ldots n=10$
2. Write out all the equations you need to evaluate with addition in between each.
3. Simplify all brackets
4. Add all terms

- $$\sum_{n=1}^{50} n = 1+2+3+\ldots+50$$

Typing that ↑ into your calculator takes too long!

1. Recognize what values of n you need to start with/end with. $n=1 \ldots n=50$
2. Write out/simplify the series you need to evaluate
3. Since it would take too long to add, use the shortcut provided.

SHORTCUT: only if function/equation $= n$
only if $n=1$

$$\boxed{\sum_{n=1}^{x} = \frac{x(x+1)}{2}} \Rightarrow \sum_{n=1}^{50} = \frac{50(50+1)}{2} = \frac{50(51)}{2} = 1275$$

WHY shortcut works:

$$\sum_{n=1}^{50} n = 1+2+3+\ldots+50$$

$$+\ \sum_{n=1}^{50} n = 50+49+48+\ldots+1$$

$$2\sum_{n=1}^{50} n = 51+51+51+\ldots+51$$

$$= 51 \times 50$$

$$\frac{2\sum_{n=1}^{50} n}{2} = \frac{51 \times 50}{2}$$

$$\sum_{n=1}^{50} n = \frac{51 \times 50}{2}$$

$$= 1275$$

EXPLICIT

nth term formulas

input (n): position of term you are looking for.
output (t_n): value of term you are looking for.

③a) Development of arithmetic n-th term formula

$$t_n = a + (n-1)d$$

term value — first term of sequence (a_1) — n is the position of the term (t_n). — constant difference (between terms of the sequence)

if a_n = position of a term then t_n = the term value in position a_n.

t_1, t_2, t_3
$\pm d \quad \pm d$

- Sequence: 5, 8, 11, 14 ... ($+3$, $+3$, $+3$)

$t_1 = 5 = 5 + (0)3 = 5$
$t_2 = 8 = 5 + (1)3 = 8$
$t_3 = 11 = 5 + (2)3 = 11$
$t_4 = 14 = 5 + (3)3 = 14$

*brackets are the simplified result of $(n-1)$ and as shown, all the statements/formulas are true.

$\therefore t_n = 5 + (n-1)3$

NOTE: constant difference (d) is only in arithmetic formula.

① Label each term value
② Identify a
↳ a = first term = 5
③ Next, move onto the bracket $(n-1)$ in the general formula. For $t_1 = 5 = 5(n-1)$, what would $(n-1)$ have to be?
↳ ZERO.
This makes sense because in this case: $(n-1) = (1-1) = 0$
④ Find d and use it as a coefficient for the bracket $(n-1)$
$d = 3$
⑤ Manipulate steps 2-4 for each term value.
⑥ Write out formula for this sequence with info. found.

EXPLICIT

③b) Development of geometric n-th term formula

$$t_n = ar^{n-1}$$

position of the term (t_n) — term value — first term of sequence (a_1) — constant ratio (between terms of the sequence)

t_1, t_2, t_3
$\cdot r \quad \cdot r$

• sequence: 5, 10, 20, 40...
$\times 2 \quad \times 2 \quad \times 2$

$t_1 = 5 = 5(2)^0 = 5$
$t_2 = 10 = 5(2)^1 = 10$
$t_3 = 20 = 5(2)^2 = 20$
$t_4 = 40 = 5(2)^3 = 40$

*(n-1) has been simplified to show all the statements/ formulas are true.

$\therefore t_n = 5(2)^{n-1}$

This makes sense because $n-1 = 1-1 = 0$

① Label each term value
② Identify a
③ Identify r
$r = 2$
④ Think about the value of $(n-1)$ that would make $t_1 = 5$
↳ $5(2)^?$
↳ $2^0 = 1$
$5(1) = 5$
⑤ Manipulate steps 2-4 for each term value.
⑥ Write out formula for this sequence with info. found

NOTE: constant ratio (r) is only in geometric formula.

③c) Arithmetic RECURSION n-th term formula

$$t_n = (t_{n-1}) + d \;,\; t_1 = \text{given}$$

term value — previous term — constant difference

• sequence: 5, 8, 11, 14
$+3 \quad +3 \quad +3$

$d = 3$
$t_1 = 5$

$\therefore t_n = (t_{n-1}) + 3,\; t_1 = 5$

① Identify variables in the general formula that you already know
② Write your general formula

Geometric RECURSION n-th term formula

$$t_n = (t_{n-1})r \;,\; t_1 = \text{given}$$

term value — previous term — constant ratio

① Repeat steps 1 and 2 from the previous example

• sequence: 5, 10, 20, 40
$\times 2 \quad \times 2 \quad \times 2$

$r = 2$
$t_1 = 5$

$\therefore t_n = (t_{n-1})2,\; t_1 = 5$

Quadratic n-th term formula

↳ If given a sequence that is quadratic, you need to "use differences to find an equation!" Detailed steps on how to do this can be found in journal # 1b of this unit.

- sequence: 4, 12, 26, 46 t_n

+8 +14 +20 Δt_n

+6 +6 $\Delta\Delta t_n$

① Find differences/ratio to determine if it is arithmetic, geometric, ...

This is quadratic. So → $t_n = an^2 + bn + c$

n	t_n	Δt_n	$\Delta\Delta t_n$
1	a+b+c	> 3a+b	> 2a
2	4a+2b+c	> 5a+b	
3	9a+3b+c		

② Solve for each unknown by following/manipulating the steps in journal #1b.

$\Delta\Delta t_n$	Δt_n	t_n
$\frac{2a}{2} = \frac{6}{2}$	$3a+b=8$	$a+b+c=4$
$\boxed{a=3}$	$3(3)+b=8$	$3+(-1)+c=4$
	$9+b=8$	$2+c=4$
	$\boxed{b=-1}$	$\boxed{c=2}$

③ After solving for your unknowns, write your n-term formula

$\therefore t_n = 3n^2 - n + 2$

Rational Sequence n-th term formula

↳ When given a rational sequence, think of the numerators and denominators as separate sequences and find their formulas separately.

$t_n = \frac{N_n}{D_n}$ ← formula for numerator sequence / ← formula for denominator sequence

- sequence: $\frac{7}{5}, \frac{16}{23}, \frac{25}{41}, \frac{34}{59}, \frac{43}{77}$

N_n

7, 16, 25, 34, 43

+9 +9 +9 +9

$N_n = a + (n-1)d$

$N_n = 7 + (n-1)9$

$\boxed{N_n = 9n - 2}$

D_n

5, 23, 41, 59, 77

+18 +18 +18 +18

$D_n = a + (n-1)d$

$D_n = 5 + (n-1)18$

$\boxed{D_n = 18n - 13}$

① Because both sequences ALONE are arithmetic, use the arithmetic explicit formula to model N_n and D_n.

② Combine the two to form formula t_n.

$\therefore t_n = \frac{9n-2}{18n-3}$

→ When creating a recursive rational sequence formula, remember to use N_n and D_n in your final answer because the recursion that occurs on the top and bottom are not the same.

General Steps for finding the n-th term formula of a sequence

① Identify if you are looking for an explicit OR recursive formula.

② Identify if your sequence is arithmetic, geometric, cubic, quadratic, or other. (Other = Fibonacci, etc.)

or, rational

③ Identify a general n-th term formula that you need to use/that you can use.

④ ~~Solve for/identify all unknown variables~~

⑤ Expand and simplify fully.

When given a middle term and you are asked to find the # of terms, remember that the same number of terms will be on either side of your middle term!

③d) Find the number of terms (Geometric Ex.)

- sequence: 0.007, 0.021, 0.063, ..., 413.343

$\cdot 3$ $\quad \cdot 3$

This is geometric, so → $t_n = a(r)^{n-1}$

$\left.\begin{array}{l} a = 0.007 \\ r = 3 \end{array}\right\} t_n = 0.007(3)^{n-1}$

$\frac{413.343}{0.007} = \frac{\cancel{0.007}(3)^{n-1}}{\cancel{0.007}}$

$59049 = (3)^{n-1}$

↕ switch to log form to solve for n

$n-1 = \log_3(59049)$
$+1 \qquad +1$

$n = \log_3(59049) + 1$

$\boxed{n = 11}$

∴ There are 11 terms in this sequence because: $413.343 = a_{11}$ and 413.343 is the LAST term.

① Classify type by finding differences/ratio

② Find the formula for this sequence

③ Because you are given the final term, sub-in for t_n and solve for n. (Because n will equal last term position and will tell you # of terms)

You can do this process when finding # of terms in an arithmetic sequence. Steps may need to be manipulated but generally, ideas stay same.

③e) Record the sum of a series in SIGMA NOTATION

(using example from ③d)

To record the sum of a series in sigma notation, you need:
→ nth term formula
→ n value series starts/ends at

- (based on ③d)
  nth term formula → $t_n = 0.007(3)^{n-1}$
  starts at $n=1$ ends at $n=11$

$$\sum_{n=1}^{11} 0.007(3)^{n-1}$$

ending n-value

starting n-value

formula

③f) Find arithmetic formula given 2 random/non-consec. terms

- $a_9 = 120$ $a_{14} = 195$

$t_n = a + (n-1)d$

$195 = a + (14-1)d$
$120 = a + (9-1)d$

↓ simplify

You have 2 unknowns now.

$195 = \cancel{a} + 13d$
$-\ 120 = \cancel{a} + 8d$

$\frac{75}{5} = \frac{5d}{5}$

$\boxed{15 = d}$

$120 = a + 8d$
$120 = a + 8(15)$
$120 = a + 120$
$-120 \quad -120$
$\boxed{0 = a}$

$\therefore t_n = (n-1)15$
$t_n = 15n - 15$

① Identify the explicit general arithmetic formula. Sub-in both terms and their n-value into the general formula.

② Subtract both formulas to eliminate "a" in both.

③ Using $d=15$, solve for "a" using a general formula with term substituted in.

④ Write out formula

Find geometric formula given 2 random/non-consec. terms

- $T_2 = 20$ $T_5 = 160$

$t_n = a(r)^{n-1}$

$160 = a(r)^{5-1}$
$20 = a(r)^{2-1}$

You have 2 unknowns now.

↓ simplify

$160 = a(r)^4$
$20 = a(r)^1$

① Identify the explicit general geometric formula. Sub-in both terms + their n value to the general formula.

$160 = \cancel{a}(r)^4$

$20 = \cancel{a}(r)^1$

$(8)^{\frac{1}{3}} = (r^3)^{\frac{1}{3}}$

$\boxed{2 = r}$

$20 = a(r)^1$

$\frac{20}{2} = \frac{a(2)^1}{2}$

$\boxed{10 = a}$

② Since the unknown variable r has exponents on it, divide both formulas and use exponent laws. (The "a" still gets eliminated in both formulas because of exponent laws).

$\therefore t_n = 10(2)^{n-1}$

$t_n = 10(2)^n(2)^{-1}$

$t_n = 10(2)^n(\frac{1}{2})$

$t_n = 5(2)^n$

③ Using r=2, solve for "a" using a general formula with a term substituted in.

④ Write out formula

④a) Development of the arithmetic sum formula

This is what an arithmetic series look like in variables:

$(a) + (a+d) + (a+2d) \ldots + n$ (nth term = last term = $a+(n-1)d$)

Note: $(a+(n-2)d)$ = term before last term

- Find the sum of all n terms in the above series

$S_n = (a) + (a+d) + (a+2d) \ldots + (a+(n-1)d)$

and so on...

$+\ S_n = (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \ldots (a+2d)$

$2S_n = (2a+(n-1)d) + (2a+(n-1)d) + (2a+(n-1)d) \ldots + (2a(n-1)d)$

① Write the series as S_n. (Which means "sum of n # of terms."

② Write the series "reverse". ★ Make sure each term on the top row has a "reverse" term on the bottom row.

→ How to add $(a+d) + (a+(n-2)d)$

$= a+d + a + dn - 2d$

$= 2a + dn - 1d$

$= 2a + (n-1)d$

③ Add both equations.

④ We see that when we add both equations, all terms are equal. So, we can re-write the right side of the equation as: $(2a+(n-1)d)n$

↑ # of terms

$\frac{2S_n}{2} = \frac{[(2a+(n-1)d)]n}{2}$

$$\boxed{S_n = \frac{(2a+(n-1)d)n}{2}}$$

This is the arithmetic sum formula.

⑤ Isolate S_n because we are looking for S_n only.

Use this to find the sum of n terms in an arithmetic series.

Another version:

$S_n = \frac{(2a+(n-1)d)n}{2}$

$\hookrightarrow S_n = \frac{(a+a+(n-1)d)n}{2}$

$S_n = \frac{n}{2}(t_1 + t_n)$

4b) Development of the geometric sum formula

This is what a geometric series looks like in variables:

$(a) + (ar) + (ar^2) \ldots + t_n$ (nth term = last term = ar^{n-1})

- Find the sum of all n terms in the series above

$$S_n = (a) + (ar) + (ar^2) \ldots + (ar^{n-1})$$
$$-\ rS_n = (ar) + (ar^2) + (ar^3) \ldots + (ar^n)$$
$$S_n - rS_n = (a) - (ar^n)$$

NOTICE: ONLY 1st term from top and last term from bottom remain.

$$\frac{S_n - rS_n}{S_n \quad S_n} = \frac{(a)}{a} - \frac{(ar^n)}{a}$$

$$\frac{S_n(1-r)}{(1-r)} = \frac{a(1-r^n)}{(1-r)}$$

① Write the series as S_n (sum of n # of terms)

② Multiply your original formula by r.

HINT → $(ar^{n-1})(r^1) = ar^{n-1+1} = ar^n$

③ Sutract both equations when you are subtracting, you notice alot of the terms cancel.

☆ When subtracting, some terms may not be written but may still exist!!!

④ Now, simplify and isolate both a and S_n.

$$\boxed{S_n = \frac{a(1-r^n)}{1-r} \overset{OR}{\leftrightarrow} S_n = \frac{a(r^n-1)}{r-1}}$$

This is the geometric sum formula.

Use this to find the sum of n terms in a geometric series.

other versions

$S_n = \frac{a(r^n-1)}{r-1}$

↳ $S_n = \frac{ar^n - a}{r-1}$

$ar^n = t_n(r)$

How?

$t_n = ar^{n-1}$

$t_n(r) = ar^{n-1}(r)^1$

$= ar^{n-1+1}$

$= ar^n$

$$S_n = \frac{t_n r - t_1}{r-1}$$

$S_n - rS_n = (a) - (ar^n)$

↳ $\frac{S_n - rS_n}{S_n \quad S_n} = (a) - (ar^n)$

$\frac{S_n(1-r)}{(1-r)} = \frac{(a) - (ar^n)}{1-r}$

$S_n = \frac{(a) - (ar^n)}{1-r}$

ar^n = term after nth term (ar^{n-1})

$ar^n = t_{n+1}$

$$S_n = \frac{t_1 - t_{n+1}}{1-r}$$

OR

$$S_n = \frac{t_{n+1} - t_1}{r-1}$$

NOTE:

By using either the arithmetic/geometric sum formula, you avoid having to do the steps shown in journal questions 4a) and 4b).

we know to use arithmetic sum formula because $5n-2$ is <u>linear</u>.

4c) Example of how to use arithmetic sum formula

$$\sum_{n=1}^{35}(5n-2) \qquad S_n = \frac{(2a+(n-1)d)n}{2}$$

① Identify S_n formula.
Look at what you need to be able to use the formula. Then, find them!
$a=?$
$d=?$
$n=?$

to find a
a = 1st term (when $n=1$)
$5(1)-2$
$5-2$
$=3$
$\boxed{a=3}$

ALWAYS PLUG-IN TO VERIFY!

to find d
you can sub-in n-values and find the d by looking at first few terms of series.

$n=1 \quad n=2 \quad n=3$
$5(1)-2 \quad 5(2)-2 \quad 5(3)-2$
$3 + 8 + 13$
$+5 \quad +5 \quad = d$
$\boxed{d=5}$

to find n
In sigma notation, n, the # of terms is ALWAYS:
(# on top − # on bottom +1)

$n = 35-1+1$
$\boxed{n=35}$

$$S_n = \frac{(2a+(n-1)d)n}{2}$$

$$S_{35} = \frac{(2(3)+(35-1)(5))35}{2}$$

$$S_{35} = \frac{(6+170)35}{2}$$

$$S_{35} = 3080$$

② Using your variable values, plug them ~~into the formula~~ and find the sum of this series.

$$\therefore \sum_{n=1}^{35}(5n-2) = 3080$$

4d) Example of how to use geometric sum formula

$$\sum_{n=1}^{9} 4(3)^{n-1} \qquad S_n = \frac{a(r^n-1)}{r-1}$$

① Identify sum formula.
Look at what you need to be able to use the formula. Then, find it!
$a=? \quad r=? \quad n=?$

we know to use geometric sum formula because $4(3)^{n-1}$ is <u>exponential</u>.

to find a
a is when $n=1$
$4(3)^{1-1}$
$4(3)^0$
$\boxed{4=a}$

to find r
compare given equation to $t_n = a(r)^{n-1}$
$\therefore \boxed{r=3}$
because: $a(r)^{n-1}$
$4(3)^{n-1}$

to find n
on top − # on bottom +1
$9-1+1 = 9$
$\boxed{n=9}$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$S_9 = \frac{4(3^9 - 1)}{3-1}$$

$$S_9 = \frac{4(19682)}{2}$$

$$S_9 = 39\,364$$

② Using the variable values that you just found, sub-in to the formula to find the sum of the series.

$$\therefore \sum_{n=1}^{9} 4(3)^{n-1} = 39\,364$$

⑥a) Simple Interest VS. compound interest

- You deposit \$1000 into your bank account that earns 5%. Show the simple (annual) interest on this amount AND show the compounded (annually) interest on this amount.

SIMPLE

$1000 \xrightarrow{\times 0.05} \boxed{50}$

Simple interest takes the original amount invested X the interest rate.

This answer tells you the dollar amount of interest your account would earn yearly.

1000) +50
1050) +50
1100) +50
1150) +50
1200) +50

So, every year, the money in your account increases by a steady amount.

Simple interest sequences/series are arithmetic. This means that their graphs are linear.

COMPOUND

$r = 0.05 + 1$
$r = 1.05$

1000) x1.05
1050) x1.05
1102.5) x1.05
1157.63

Compound interest takes the current TOTAL balance in your account X the interest rate. So, every year, your interest would grow because the balance in your account would continue to grow.

Compound interest sequences/series are geometric. This means that their graphs are exponential.

⑥b) Simple Interest formulas and how to use them

$$A = P + Prt$$

- A: final amount
- P: principal (original amount)
- r: rate (interest rate as a decimal)
- t: time (IN YEARS)

TRANSLATING TIME TO YEARS:
→ days $= \frac{x}{365}$ → weeks $= \frac{x}{52}$
→ months $= \frac{x}{12}$

In the above formula, you are taking your principal amount and adding it to the product of your principal X interest rate X time.

$$I = Prt$$

I: Interest (amount in \$)

You can use this formula for a simple interest question if you are only being asked to find the total interest on a principal after t years.

*when being asked to solve for r, remember to x100 to represent your answer as a percent.

- If you invest \$3500 in a savings account that pays 4%, simple interest, how much interest will you earn after 38 months? How much money will you have in your savings account.

$A = P + Prt$

$P = 3500$

$r = 4\% = 0.04$

$t = 38$ months

$t = \frac{38}{12} = \frac{19}{6}$

② Find all the values for your variables then solve for A.

① The question asks you to ~~do 2 things. It asks you to~~ find the interest as an amount ~~and the account balance~~ after 3 years. So, you can use $A = P + Prt$ to find the balance in the savings account. Then, if you subtract your principal from that amount, you will have the amount of interest earned.

$A = 3500 + 3500(0.04)(\frac{19}{6})$

$A = 3943.33$

∴ Your account balance after 38 months is \$3943.33

$A - P = Prt$ $\boxed{\text{OR}}$ $A - P = I$ ← interest (amount in \$)

$3943.33 - 3500 = I$

$443.33 = I$

∴ The interest you earned on your account after 38 months was \$443.33

*NOTE: This question can be done vice versa by first finding the amount of interest you will earn. Then, add it to your principal.

6c) Compound Interest formulas and how to use them

$$A = P(1+i)^n$$

n ← total compounding periods

A: final amount

P: principal

i: periodic rate (decimal). This means that you must take the given interest rate and divide it by the # of times your principal will be compounded IN A YEAR.

$$n = Ct$$

this means take the # of times your rate will be compounded in a year and multiply it by your time in years.

$$i = \frac{r}{C}$$

i: periodic rate

r ← annual rate

C ← # of times compounded in a year

TRANSLATING COMPOUNDING PERIODS (C)

annually = 1 semi-anually = 2 weekly = 52

quarterly = 4 monthly = 12 daily = 365

*Before you begin to use the compounding interest formula, you need to solve for i and n.

*If you are asked for the annual interest rate, remember to solve for i then plug back into $i = \frac{r}{c}$ formula to solve for r. After solving for r, remember to convert it into a percent.

- If you want to be able to put a $100,000 deposit down on a house when you are 25, how much money should you put in the bank now as an 18-year old? You are able to get a 5.8% compounded annually interest rate.

$A = P(1+i)^n \rightarrow i = \frac{r}{c} \quad n = Ct$

$A = 100\,000$

$r = 5.8\% = 0.058$

$C = \text{annually} = 1$

$i = \frac{0.058}{1}$

$i = 0.058$

$t = 25 - 18 = 7$

$n = (1)(7)$

$n = 7$

$P = ?$

$\frac{100\,000}{(1+0.058)^7} = \frac{P(1+0.058)^7}{(1+0.058)^7}$

$\frac{100\,000}{(1.058)^7} = P$

$67\,390.76 = P$

① Recognize what values for variables the question gives you as well as what you need to find. Use the additional (i) and (n) formulas to solve for them.

② Plug-in all the information that you know. Then, isolate for P to find your principal.

∴ Now, as an 18 yr old, you have to put in $67 390.76 to have $100,000 at 25 yrs old.

- If you were able to get the above account (↑) compounded semi-annually instead of annually, how much money would you save?

$A = 100\,000$

$r = 0.058$

$C = \text{semiannually} = 2$

$i = \frac{0.058}{2}$

$i = 0.029$

$t = 7$

$n = (2)(7)$

$n = 14$

$P = ?$

$\frac{100000}{(1+0.029)^{14}} = \frac{P(1+0.029)^{14}}{(1+0.029)^{14}}$

$\frac{100\,000}{(1.029)^{14}} = P$

$67016.96 = P$

① The only variables that change in this case are C, i, and n. Find these new values

② Find P now when your interest is being compounded semi-annually.

∴ Now, as an 18 yr old, you would only have to put in $67016.96 as an 18 yr old. So, you would have to deposit $373.80 less with this option.

$$\begin{array}{r} 67390.76 \\ -\ 67016.96 \\ \hline 373.80 \end{array}$$

☆ if being asked to solve for r, remember to re-write it as a percent!

⑥d) Continuous Compounding Interest formula and how to use it

$A = Pe^{rt}$ ← time (IN YEARS)

A ↑ Final value

P ↑ principal

r ↖ rate (interest rate as decimal)

e = constant

$e \doteq 2.718$

- If you invest \$3,200 for 4 years at 6% continuous interest, what is the ending balance?

①Identify all variable values given to you and what you need to find.

$A = Pe^{rt}$

$P = 3200$

$t = 4$

$r = 6\% = 0.06$

$A = ?$

②Plug-in information you know and solve for A.

$A = 3200e^{0.06(4)}$

$A = 3200e^{0.24}$

$A \doteq 4067.997$

$A \doteq 4068$

∴ After 4 years, the ending balance would be \$4068.00

- If you invested \$3933.14 into an account with continuous interest that, in 3 years, gave you \$5000, what is your continuous interest rate?

①Identify given variables and identify what you are looking for.

$A = Pe^{rt}$

$P = 3933.14$

$A = 5000$

$t = 3$

$r = ?$

②Solve for r

$\frac{5000}{3933.14} = \frac{\cancel{3933.14}e^{r(3)}}{\cancel{3933.14}}$

$\frac{5000}{3933.14} = e^{3r}$ ← isolate r

$\frac{5000}{3933.14} = (e^3)^r$ ← switch forms

↕

$r = \log_{e^3}\left(\frac{5000}{3933.14}\right)$

$r \doteq 0.0799 \times 100$

$r \doteq 7.99$

$r \doteq 8$

∴ The continuous interest rate on your investment would be 8%.

7 a) Annuities - These are recurring investments. Meaning that unlike the investments spoken about in simple/compound interest, there is NO principal amount. With annuities, there are multiple regular deposits that occur for a period of time.

KEY WORDS → "each year" "every month"

Also, all deposits made are assumed to be at the end of each time period unless specified.

ie: "deposited every month" = the deposit is made at the end of each month.

Coming Up With FV (FUTURE VALUE) Formula

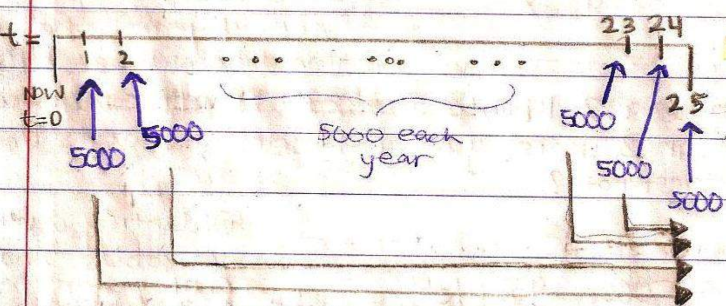
With this formula you want to know how much will a number of investments be worth at the end of a certain time period.

- If you invest $5000 a year into a trust account, then how much will be in the account after 25 years if the account earns interest at a rate of 8.5% per year compounded yearly?

$r = 8.5\% = 0.085 = \frac{0.085}{1}$

$\therefore r = 1 + 0.085 = 1.085$

① Because your interest rate is being compounded, find the periodic rate. (i)



② Draw a timeline and label the time periods.

③ Identify when the regular deposits are made.

④ Look at each deposit and map out how long they are in the account.

⑤ How long each deposit has been in the bank determines how much interest it receives. Think of each deposit as single investments receiving their own interest. And, the future value is just the sum of all these individual investments/deposits.

geometric series

t=1 → 24 years t=2 → 23 years t=24 → 1 year t=25 → 0

$\underbrace{5000(1.085)^{24}}_{P(1+i)^n} + 5000(1.085)^{23} \dots + 5000(1.085) + 5000$

⑥ To find your FV, you need the sum of your series from above.

$S_n = \frac{a(r^n - 1)}{r-1}$

$n = 25$ $r = 1.085$

$a = 5000$

$S_{25} = \frac{5000(1.085^{25} - 1)}{1.085 - 1}$

$FV = R\left[\frac{(1+i)^n - 1}{i}\right]$

$\therefore S_n$ or FV will be $393 338.96

☆ When a value's interest is being DISCOUNTED, the total compounding period exponent is negative

Coming up with PV (PRESENT VALUE) formula

With this formula you want to know how much money you could get right now. So, you want to know the value of your regular deposits DISCOUNTED (without the interest rate) at the present time.

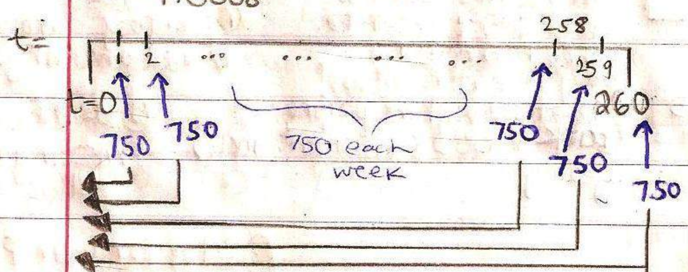
- A lottery offers a prize of $750 every week for 5 years. The first payment will be made 1 week from now. If money can be invested at 4.4% per year compounded weekly, what cash payment could be received immediately?

$r = 4.4\% = \frac{0.044}{52} \doteq 0.0008$

$1 + 0.0008 = r$

$\therefore r \doteq 1.0008$

① Because your interest rate is being compounded weekly, find the periodic rate. (i)



② Draw a timeline and label the time periods.

③ Identify when the regular deposits are made.

④ Look at each deposit and map out how long/how many periods would it have to pass to be discounted to t=0.

⑤ How long would it take for each weekly prize to be discounted? Think of each prize as a single investment losing the interest it would normally have received. The present value is just the sum of all the discounted cash prizes.

geometric series

t=1 → -1 week t=2 → -2 weeks t=259 → -259 weeks t=260 → -260 weeks

$$\underbrace{750(1.0008)^{-1}}_{P(1+i)^n} + 750(1.0008)^{-2} \dots + 750(1.0008)^{-259} + 750(1.0008)^{-260}$$

⑥ To find your PV, you have to find the sum of your series from above.

$S_n = \frac{a(r^n - 1)}{r-1}$

$$S_{260} = \frac{750(1.0008)^{-260}(1.0008^{260} - 1)}{1.0008 - 1}$$

$n = 260$

$a = 750(1.0008)^{-260}$

$r = 1.0008$

(Distribute and simplify $(1.0008)^{-260}$)

$$S_{260} = 750\,\frac{(1.0008)^{-260}(1.0008)^{260} - (1.0008)^{-260}}{0.0008}$$

$\therefore S_n$ or PV will be $175 992.58

$$S_{260} = 750\,\frac{(1 - (1.0008)^{-260})}{0.0008}$$

$$PV = R\left[\frac{1 - (1+i)^{-n}}{i}\right]$$

Interest only:

Take: FV − (R × # of deposits)

$1-(1+i)^{-n}$ discounts the interest of R.

FUTURE VALUE ANNUITY FORMULA

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

FV = future value, R = regular deposit, $i = \frac{r}{c}$, $n = ct$

PRESENT VALUE ANNUITY FORMULA

$$PV = \frac{R[1-(1+i)^{-n}]}{i}$$

PV = present value, R = regular deposit, $i = \frac{r}{c}$, $n = ct$

7b) Finding PV in a present value annuity

- AmericanGeneral offers a 10-year annuity with a rate of 6.65% compounded annually. How much should you pay for one of these annuities if you want to receive payments of $5,000 anually over the 10 year period?

$$PV = \frac{R[1-(1+i)^{-n}]}{i}$$

We use this formula because we want to know how much to pay for it now so that we will be paying back $5000 (with interest incl.) yearly.

① Identify formula to be used and why.

$t = 10$
$r = 6.65\%$ $\boxed{i = 0.0665}$
$c = 1$ $\boxed{n = 10}$
$PV = ?$
$\boxed{R = 5000}$

② Identify given/missing variable values

③ Solve for your PV

$$PV = \frac{5000[1-(1+0.0665)^{-10}]}{0.0665}$$

$$PV = 35\,693.18$$

④ Write ∴ statement

∴ You should pay $35,693.18 for one of these annuities.

7c) Finding n in a future value annuity

- How many payments must occur in an annuity that has monthly payments of $340 and an annual interest rate of 6% compounded monthly to give a total value of $20,000?

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

We use this formula because we know our balance of the account once our monthly payments have accumulated interested. So, our balance is FV and we need to find # of payments.

① Identify formula to be used and why.

$\boxed{FV = 20000}$
$\boxed{R = 340}$ $\boxed{i = 0.005}$
$r = 6\%$
$c = 12$
$n = ?$

② Identify given/missing variable values.

$$\frac{20000}{1} = \frac{340[(1+0.005)^n - 1]}{0.005}$$

③ Solve for your n

④ Write ∴ statement

$$\frac{20000(0.005)}{340} + 1 = (1.005)^n$$

$$\updownarrow$$

$$n = \log_{1.005}\left(\frac{20000(0.005)}{340} + 1\right) \qquad \boxed{n = 51.69}$$

∴ After about 51.69 payments, this annuity will have a balance of $20 000.

With mortgages, the compounding frequency and the payment frequency usually do not match. So, you can't just use a PV formula.

7d) How to do mortgage calculations

How to change the interest rate of the compounding period and make it equivalent to the payment frequency

- Bob just purchased a home for $250,000. He was able to make a $50,000 deposit but had to mortgage the rest at a rate of 5% compounded semi-annually that will be paid off (amoritized) for 25 years.

General formula for converting interest rate: $(1+s)^2 = (1+m)^{12}$

☆ This is a semi-annual to monthly conversion ☆

① Take the annual interest rate of 5% and find the periodic rate.

$r = 5\%$ $\quad i = \frac{r}{c}$ $\quad i = \frac{0.05}{2} = \boxed{0.025}$

$0.025 = s$ (your semi-annual rate)

② To find your monthly rate, simply isolate m. $\quad (1+0.025)^2 = (1+m)^{12}$

$$\sqrt[12]{(1+0.025)^2} = \sqrt[12]{(1+m)^{12}}$$

$$\sqrt[12]{(1.025)^2} = 1+m$$
$$-1 \qquad -1$$

$$\sqrt[12]{(1.025)^2} - 1 = m \leftarrow \text{rate as decimal}$$

This is your converted monthly interest rate.

$$\boxed{0.004123915 = m}$$

x100

$$\boxed{0.412391546\% = m}$$

DO NOT ROUND m!!!

How to find the regular monthly mortgage payment

☆ TO DO THIS: you need the periodic rate to be converted to monthly. Since we already did this, we can use **m**.

$$PV = \frac{R[1-(1+m)^{-n}]}{m}$$

$PV = 250000 - 50000$
(cost) − (deposit)

$\boxed{PV = 200000}$
(mortgage)

① Use the PV formula because the mortgage that you borrow/use to buy a house is your PV. It has ~~no interest~~.

$R = ?$

$\boxed{m = 0.004123915...}$

$n = ct$ $\qquad c = 12$ (NOW)

$n = 12(25)$

$\boxed{n = 300}$

② Identify given/missing variable values

$$\frac{200\,000}{1} = \frac{R[1-(1+0.004123915)^{-300}]}{0.004123915}$$

③ Solve for R

④ Write ∴ statement

$$\frac{200\,000(0.004123915)}{[1-(1.004123915)^{-300}]} = R \qquad \boxed{1163.21 = R}$$

∴ The regular monthly mortgage payment will be \$1163.21.

How to find total amount paid for house

\$1163.21 x 300 = \$348 962.97

↑ n = ct, n = 300

+ \$50,000 ← DOWN PAYMENT!

∴ total amount paid = $\underbrace{\$398\,962.97}$

This is how much money (with interest on the mortgage) your house cost you.

① To find how much you paid for the house, you don't look at your mortgage amount alone. You multiply your regular payments by the number of times you made them (total compounding periods)

How to find what amount of interest you paid

\$398962.97 − \$250000
= \$148962.97

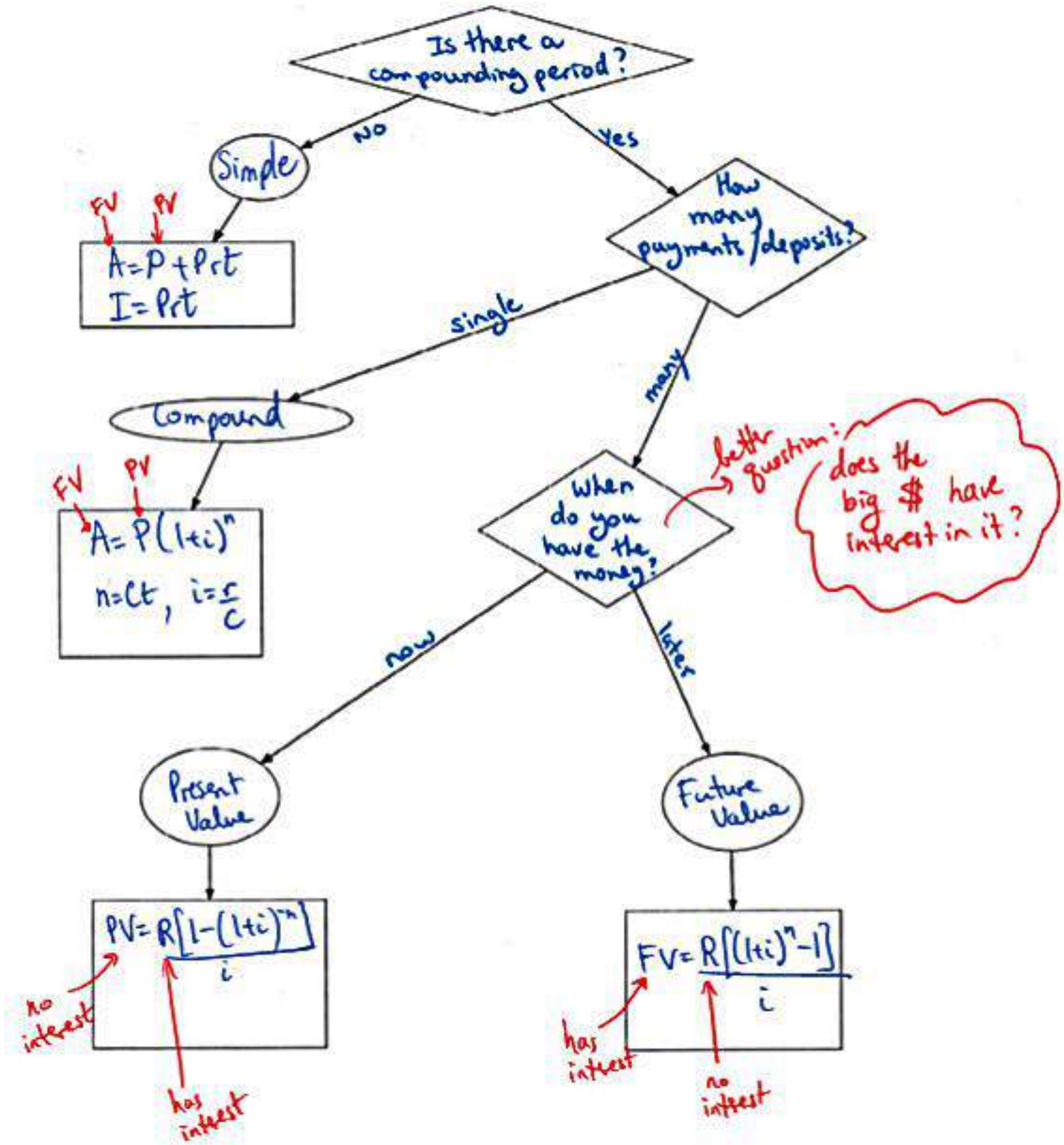
OR

\$1163.21 x 300 = \$348962.97
mortgage = \$200 000
\$348962.97 − \$200000
= \$148962.97

① To find how much interest you paid, take the actual price of the home and subtract it by how much you paid for the house.

② OR, you can take the # of your mortgage payments x your monthly payment and compare it to your mortgage.

Finance Word Problem Chart → What formula to use and when



Note: PV is usually loans and mortgages