

Piecewise Function: A function whose rule changes depending on the value of the input

Piecewise Functions

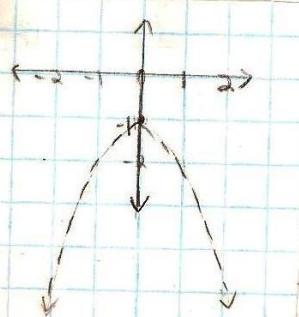
⑧a) How to sketch a piecewise function

- Graph.

$$f(x) = \begin{cases} -x^2 - 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$$

① Begin by graphing the entire function $f(x) = -x^2 - 1$ (graph it faintly or use dotted lines)

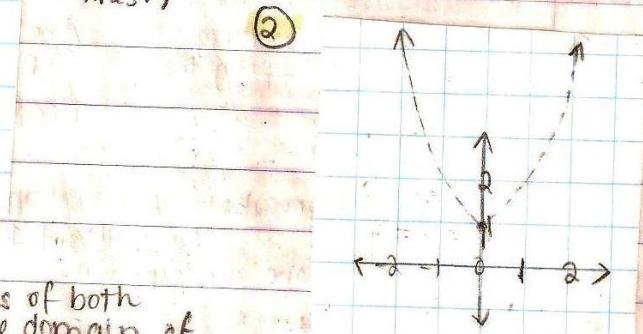
①



② Then, graph the entire function

$$f(x) = x^2 + 1 \quad (\text{graph it faintly or use dotted lines.})$$

②



③

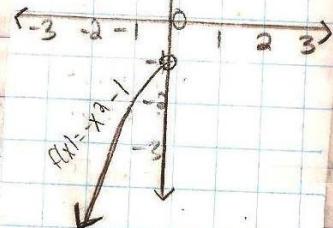
Looking at the sketches of both functions, define the domain of each and graph the entire piecewise function on a grid. (When $<$ or $>$ is used, use a \circ on the line. When \leq or \geq is used, use a \bullet on the line.)

So... for $f(x) = -x^2 - 1$, only graph points where x is less than 0.

for $f(x) = x^2 + 1$, only graph points where x is greater than or equal to 0.

③

NOTE: because in $f(x) = -x^2 - 1$
0 was not included as an x -value, an open circle was used.

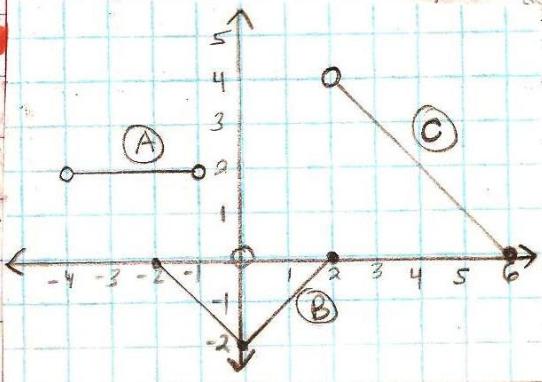


shortcut!

To sketch piecewise functions faster, skip steps 1 and 2 and go straight to step 3.

Instead of drawing out the function and then defining the domain, you can do it all at once.

⑧b) Finding the equations of a piecewise function



① Identify the "pieces" (functions) you need to determine equations for. Then, identify their parent functions.

Ⓐ parent function:

$$f(x) = c$$

Ⓑ parent function:

$$f(x) = |x|$$

Ⓒ parent function: $f(x) = x$

② Identify transformations that have been applied to the parent function. Use this to help create equations for each piece. Then, define the domain for each equation.

Ⓐ $f(x) = c$
 $f(x) = 2$

if: $-4 \leq x \leq -1$

Ⓑ $f(x) = |x|$
 $f(x) = |x| - 2$

if: $-2 \leq x \leq 2$

Ⓒ $f(x) = x$

$f(x) = mx + b$

$$m = \frac{\Delta y}{\Delta x} = \frac{0-4}{6-2} = \frac{-4}{4}$$

$$m = -1$$

$$\begin{aligned} y &= mx + b \\ 0 &= -6 + b \\ 0 &= -6 + b \end{aligned}$$

$f(x) = -x + 6$
 if: $2 \leq x \leq 6$

③ Write out your entire completed piecewise function.

$$f(x) = \begin{cases} 2 & \text{if } -4 \leq x \leq -1 \\ |x| - 2 & \text{if } -2 \leq x \leq 2 \\ -x + 6 & \text{if } 2 \leq x \leq 6 \end{cases}$$

⑧c) piecewise Function word problem

- Maya runs a 109-km triathlon. She swims 2 km in 1 hour, then bikes 92 km in 4 hours and finally runs 15 km in 3 hours. Sketch a graph of Maya's distance versus time and then write a piecewise function.

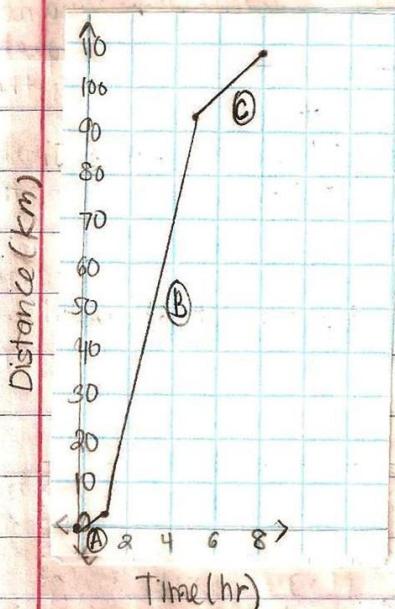
Time	Distance (travelled)
1hr	2 km
4hr	92 km
3hr	15 km

① Create a table of values using the information given to you.

x, y
time, distance

6

- (0,0) \rightarrow (1,2) \rightarrow (5,94) \rightarrow (8,109)
- +1hr +4hr +3hr
+2km +92km +15km
- ② Create points to plot onto a grid.
(Look at the data as a whole!)



③ Create a graph by plotting your points. Then, identify the "pieces" (functions) you need to determine equations for.

④ Begin to determine/create equations for each "piece". Start by identifying the parent function. Then, define the domain for each.

① parent function: $f(x) = x$

$$m = \frac{\Delta y}{\Delta x} = \frac{2-0}{1-0} = 2$$

$$m = 2$$

$$y = mx + b$$

$$0 = 2(0) + b$$

$$0 = b$$

$$f(x) = 2x$$

if: $0 \leq x \leq 1$

② parent function: $f(x) = x$

$$m = \frac{\Delta y}{\Delta x} = \frac{109-94}{8-5} = \frac{15}{3}$$

$$m = 5$$

$$y = mx + b$$

$$109 = 5(8) + b$$

$$109 - 40 = b$$

$$69 = b$$

$$f(x) = 5x + 69$$

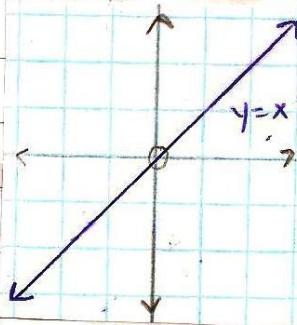
if: $5 \leq x \leq 8$

⑤ Write out the completed piecewise function.

$$f(x) = \begin{cases} 2x & \text{if: } 0 \leq x \leq 1 \\ 23x - 21 & \text{if: } 1 < x \leq 5 \\ 5x + 69 & \text{if: } 5 \leq x \leq 8 \end{cases}$$

⑧ d) Absolute Value functions as piecewise functions

- Sketch $y = |x|$



① Graph the original function $y = x$

② Since you want the absolute value of x , all output values should be positive (even if the input value is negative).

To "apply" the absolute value to all input values, re-write the function $y = |x|$. You will have 2 new functions that together, form a piecewise function.

$$y = |x|$$

if this is positive, the output will be x .

so: $f(x) = x \text{ if } x \geq 0$

$$y = |x|$$

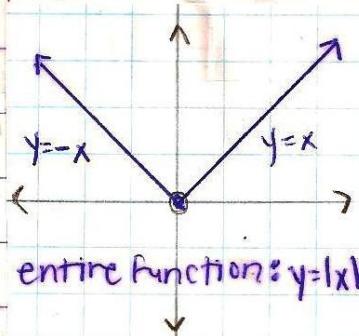
if this is negative, the output will be the opposite of x . To get the opposite, multiply by -1 .

so: $f(x) = -x \text{ if } x < 0$

P

The process that was just completed is finding the piecewise function of an absolute value function ALGEBRAICALLY.

- ③ To find the piecewise function of the absolute value function $y = |x|$ graphically, reflect all points below the x -axis in the x -axis to make their outputs (y -values) positive.



The result of this absolute value function in piecewise notation is:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Find the piecewise representation of $y = |2x - 4|$ without sketching until the end.

$$y = |2x - 4|$$

↑
if positive:

$$f(x) = 2x - 4 \text{ if } 2x - 4 \geq 0$$

$$+4 +4$$

$$\frac{2x}{2} \geq \frac{4}{2}$$

$$x \geq 2$$

$$y = |2x - 4|$$

↑
if negative:

$$f(x) = -(2x - 4) \text{ if } 2x - 4 < 0$$

$$(-2x + 4) \rightarrow +4 +4$$

$$\frac{2x}{2} < \frac{4}{2}$$

$$x < 2$$

① Find the 2 functions that make up the piecewise function of $y = |2x - 4|$

② Simplify function and solve each inequality (isolate x)

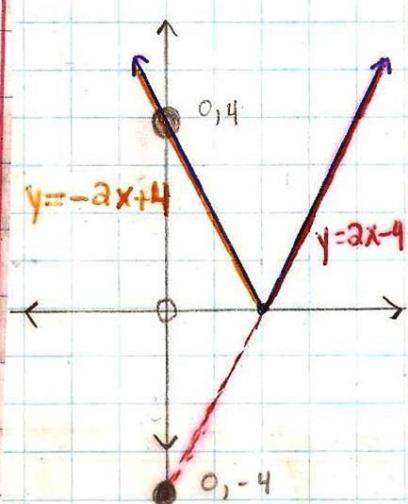
$$\text{So: } f(x) = 2x - 4 \text{ if: } x \geq 2$$

$$\text{So: } f(x) = -2x + 4 \text{ if: } x < 2$$

$$f(x) = \begin{cases} 2x - 4 & \text{if: } x \geq 2 \\ -2x + 4 & \text{if: } x < 2 \end{cases}$$

③ Rewrite using piecewise notation

④ Sketch $y = |2x - 4|$ to verify your answer.



\curvearrowleft is the function $y = |2x - 4|$

If we look at each line separately, we can see they are the functions $y = 2x - 4$ and $y = -2x + 4$ that we algebraically found in steps 1-3. We can see that they have the same slope and y-intercept so, we know we got our answer right.

- Find the piecewise representation of $y = |x^2 - 4|$ without sketching until the end.

$x^2 - 4$ factors to: $(x+2)(x-2)$

$$x+2=0 \quad x-2=0$$

$$x = -2 \quad x = 2$$

① Because this is a quadratic absolute value function, we should factor to find the zeros because the zeros are points on the function that don't change even after applying the absolute value to them.

$$y = |x^2 - 4|$$

↑
if positive:

$$f(x) = x^2 - 4 \text{ if } x \geq 2$$

AND

$$(f(x) = x^2 - 4)$$

if $x \leq -2$

② By looking at the function $x^2 - 4$, we can see that it is a parabola that has been shifted down 4 units and opens up.

We can also tell that the points where $x < -2$ or $x > 2$ are above the x -axis since -2 and 2 are our zeros.

$$y = |x^2 - 4|$$

↑
if negative

$$(f(x) = -(x^2 - 4) \text{ if } -2 < x < 2)$$

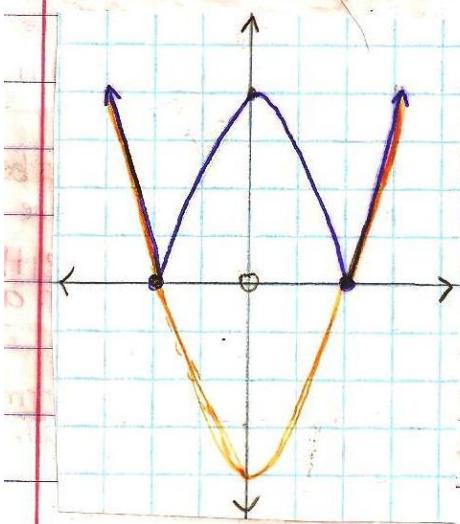
multiply by -1
to reflect in
 x -axis

$$f(x) = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \\ x^2 - 4 & \text{if } x \geq 2 \\ -x^2 + 4 & \text{if } -2 < x < 2 \end{cases}$$

③ Define the domain for the remainder of the function that you know is below the zeros.

* Not only do you have to define its domain, but you also have to transform it so that it's not below your zeros.

④ If needed, simplify all your functions and then record the piecewise function of $y = |x^2 - 4|$ using piecewise notation.



is the function $y = |x^2 - 4|$

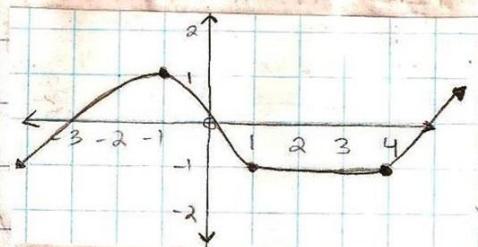
is the function $y = x^2 - 4$

as you can see, we did get the right answer by algebraically

finding the piecewise function for $y = |x^2 - 4|$ because we kept the parts of the original function that already had positive outputs and we reflected the part of the original function that had negative outputs (reflected it in the x -axis)

Properties of Functions

(9a) Increasing, Decreasing, Constant



① Look for and identify parts of the graph that are increasing, decreasing, or staying constant.
★ Read the graph from left to right

② To record the parts of the function that are increasing, use interval notation to state for which values of x , the graph is increasing.

$(-\infty, -1)$ ← note: at -1 , the graph is neither increasing or decreasing so, use a round bracket.

$(4, \infty)$
↑
in this case, you have 2 intervals where the function is increasing. So, create a union between both intervals.

$(-\infty, -1) \cup (4, \infty)$

③ To record the parts of the function that are decreasing, use interval notation to state for which values of x , the graph is decreasing.

$(-1, 1)$

④ To record the parts of the graph that are staying constant, use interval notation to state for which values of x , the graph remains constant.

$(1, 4)$

★ Intervals for increasing, decreasing, and constant are written to ONLY correspond with x -values in each increasing, decreasing, or constant part.

(9b) Monotonicity: the tendency of a function to increase or decrease over its range.

Monotonicity helps determine if there is a unique inverse because if a function is strictly monotonic, then it will always have an inverse.

Strictly Monotonic Function: a function that is either always increasing or always decreasing.

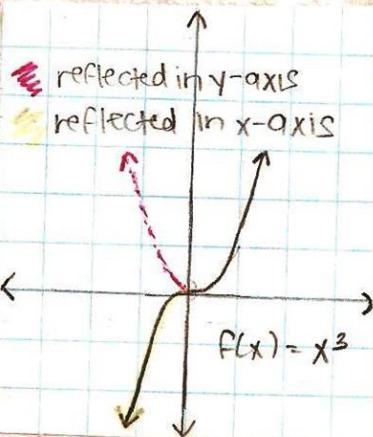
This property of strictly monotonic functions helps us determine if functions (unique or standard) have inverses. This is made possible because all strictly monotonic functions are one-to-one functions.

⑨ c) Odd, Even, Neither

Odd Functions \rightarrow (symmetry definition) everything that is right of the y-axis is reflected once in the y-axis and then reflected once in the x-axis.

In order for a function to be an odd function:

$$f(x) = -f(-x) \quad \text{OR} \rightarrow -f(x) = f(-x)$$



• $f(x) = x^3$

points on graph: $(2, 8)$ points on $(3, 27)$

$(-2, -8)$ graph: $(-3, -27)$

$$f(x) = -F(-x)$$

$$-f(x) = F(-x)$$

$$f(2) = -f(-2)$$

$$-f(3) = F(-3)$$

$$8 = 8$$

$$-27 = -27$$

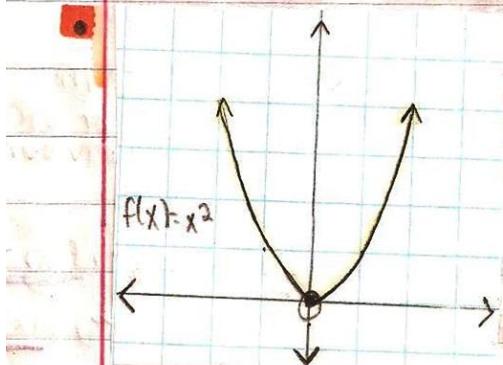
∴ as proven graphically and algebraically, $f(x) = x^3$ is an odd function.

Note: $f(x) = -f(-x)$ says that for opposite x-values of an odd function, the opposite of $f(-x)$ will be the same as $f(x)$.

Even Functions \rightarrow (symmetry definition) the function is symmetrical about the y-axis.

In order for a function to be an even function:

$$f(x) = f(-x)$$



• $f(x) = x^2$

points on graph: $(2, 4)$

$(-2, 4)$

$$f(x) = F(-x)$$

$$f(2) = F(-2)$$

$$4 = 4$$

✓

∴ as

proven
graphically

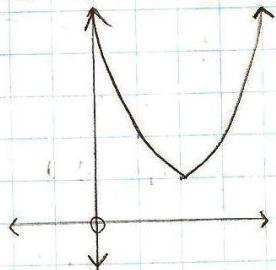
and algebraically,
 $f(x) = x^2$ is an odd
function.

note:

$f(x) = f(-x)$ says
that the output for opposite
x-values will be the same.

Neither → it is possible for functions to be neither odd or even. This occurs when the function doesn't meet the criteria of an odd or even function.

$$f(x) = (x-2)^2 + 1$$



this parabola isn't symmetrical about the y-axis like the one in the previous example. So, it isn't an even function.

this parabola also isn't a function where everything to the right of the y-axis has been reflected once in the y-axis and then once more in the x-axis. So, it isn't an odd function either.

④ $f(x) = (x-2)^2 + 1$

points on the graph: $(1, 2)$ $(-1, 10)$
 $(3, 2)$ $(-3, 26)$

is it odd?

$$f(x) = -f(-x)$$

$$f(1) = -f(-1)$$

$$2 \neq -10$$

is it even?

$$f(x) = f(-x)$$

$$f(3) = f(-3)$$

$$2 = 26$$

X

∴ as proven graphically and algebraically, $f(x) = (x-2)^2 + 1$ is neither an odd or even function.

X

⑨(d) End Behaviour of Functions

end behaviour → describes the y-values of a function as the x-values approach $-\infty$ and ∞ .

★ GENERAL NOTATION ★

You can replace $\infty, -\infty$ with numbers if needed.

As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$ or ∞ (choose one)

As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$ or ∞ (choose one)

ALWAYS use this notation for stating the end behaviour of functions.

For this graph, as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

For this graph, as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$

NOTE: to use this notation, look at which direction the y-values are moving towards in regards to the direction of x-values.

