

③ F) SOAP → same, opposite, always positive

MHF topic

↓ ↗ if you common factor something out,
just write it beside $(x+y)$... remember to include it!!

③ F) Factor a sum of cubes $(x^3 + y^3)$

To factor a sum of cubes, use this general formula:

$$(x+y)(x^2 - xy + y^2)$$

$$64x^3 + 125$$

$$(4x)^3 + (5)^3$$

$$\begin{matrix} \swarrow & \searrow \\ x & y \end{matrix}$$

① Rewrite each term as its cube root being cubed.

② Substitute the information you have into the general factored form formula.

$$(x+y)(x^2 - xy + y^2)$$

$$(4x+5)((4x)^2 - (4x)(5) + (5)^2)$$

$$(4x+5)(16x^2 - 20x + 25)$$

② Simplify

$$(4x+5)(16x^2 - 20x + 25)$$

Factor a difference of cubes $(x^3 - y^3)$

To factor a difference of cubes, use this general formula:

$$(x-y)(x^2 + xy + y^2)$$

$$27y^3 - 8$$

$$(3y)^3 - (2)^3$$

$$\begin{matrix} \swarrow & \searrow \\ x & y \end{matrix}$$

① Rewrite each term as its cube root being cubed.

② Substitute the information you have into the general factored form formula.

$$(x-y)(x^2 + xy + y^2)$$

$$(3y-2)((3y)^2 + 3y)(2) + (2)^2)$$

$$(3y-2)(9y^2 + 6y + 4)$$

ALWAYS
factor out
lowest exponent

③ g) How to factor smallest exponent out

$$(3x^3 + 1) \cdot 4(-4x^2 - 3)^3 \cdot -8x + (-4x^2 - 3)^4 \cdot 9x^2$$

$$(3x^3 + 1) \cancel{(-4x^2 - 3)^3} \cancel{(-32x)} + (-4x^2 - 3)^4 (9x^2)$$

$$GCF = (-4x^2 - 3)^3 (x)$$

① Divide expression at the addition sign

$$(-4x^2 - 3)^3 (x) [(3x^3 + 1)(-32)] + [(-4x^2 - 3)(9x^2)]$$

② Multiply coefficients on "either side".

$$(-4x^2 - 3)^2 (x) (-96x^3 - 32 - 36x^3 - 27x)$$

③ Find a GCF from either side and factor it out.

$$(-4x^2 - 3)^2 (x) (-132x^3 - 27x - 32)$$

④ Write remaining terms in brackets and simplify them to form one expression.

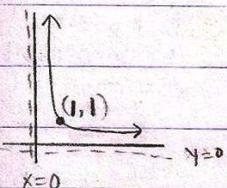
Unit 6 - Polynomials

(1a) What is a power function? $y = x^a$

Four cases that are possible for the right side of the graph

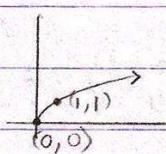
$a < 0$

$$y = \frac{1}{x} = x^{-1}$$



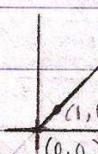
$0 < a < 1$

$$y = \sqrt{x} = x^{\frac{1}{2}}$$



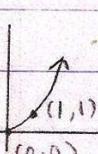
$a = 1$

$$y = x^1$$



$a > 1$

$$y = x^2$$



How to decide what the left side would look like:

If x is under any even root, there is no left side.

$\begin{cases} \text{LS} \\ \text{if } a \text{ odd} \\ \text{even} \end{cases}$

$$\text{i.e. } x^{\frac{1}{2}}, \sqrt{x}, x^{\frac{2}{3}}$$

(even roots can't have 0 inputs)

Determine the type of symmetry by using:

$$\begin{cases} f(x) \text{ even} \\ f(-x) \text{ even} \\ -f(x) \text{ odd} \end{cases}$$

1st: draw your right side and apply reflections if needed.

2nd: use three formulas on left to determine where to draw left side.

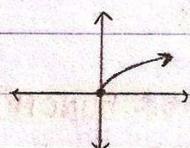
NOTE: If there is a reflection, reflect right side first then based on symmetry, draw left side.

• $k(x) = 4x^{\frac{1}{4}}$

how would this graph look?

$$a = \frac{1}{4} \therefore 0 < a < 1$$

no reflections



① Identify a

② Look for any reflections in the x -axis. Then, draw based on sketches above.

③ To determine if there is a LS, look for any roots.

• $m(x) = -5x^{\frac{2}{3}}$

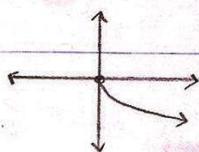
how would this graph look?

(refer to steps for previous example, those scenarios are identical)

*except this function has a reflection!

$$x^{\frac{1}{3}} = \sqrt[3]{x} = \text{even root}$$

$\therefore \text{NO LEFT SIDE!}$



(no left side because of even root)

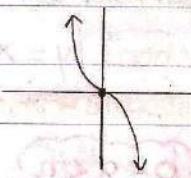
$\bullet b(x) = -7x^{\frac{5}{3}}$ or $-7\sqrt[3]{x^5}$

$$a = \frac{5}{3} \therefore a > 1$$

reflection

$$f(x) = -7\sqrt[3]{x^5}$$

$$\begin{aligned} f(-x) &= -7\sqrt[3]{(-x)^5} & -f(x) &= (-1)(-7\sqrt[3]{x^5}) \\ &= -7\sqrt[3]{-x^5} & &= 7\sqrt[3]{x^5} \\ &= (-1)(-7\sqrt[3]{x^5}) \\ &= 7\sqrt[3]{x^5} \end{aligned}$$



① Identify a.

② Identify any reflections and then sketch right side.

③ To determine LS, look for any even roots.

④ If no even roots, determine the type of symmetry present.

∴ Since $f(-x)$ and $-f(x)$ are equal, the function has odd symmetry.

$\bullet r(x) = 5x^{-2}$ or $\frac{5}{x^2}$

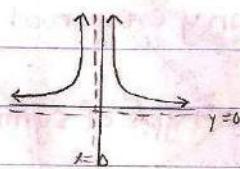
① Identify a. $a = -2 \therefore a < 0$

② Identify any reflections and draw the right side

③ To determine LS, look for even roots.

④ If no even roots, determine type of symmetry the function has.

$$f(x) = \frac{5}{x^2} \quad f(-x) = \frac{5}{(-x)^2} \quad -f(x) = \frac{-5}{x^2}$$



⑤ Use odd symmetry to reflect right side to draw/come up with left side.

∴ Since $f(x)$ and $f(-x)$ are equal, this function has even symmetry.

⑤ Use even symmetry to reflect right side to draw/come up with left side.

$$\frac{5}{x^2}$$

COMPARE: EVEN POWER FUNCTIONS VS. ODD POWER FUNCTIONS

→ Integer power functions ONLY

Even power functions: x^2, x^4, x^6, \dots all have even symmetry (regardless of reflections!)

Odd power functions: x, x^3, x^5, \dots all have odd symmetry (regardless of reflections!)

Refer to Unit 1 List B sketches! :)

BOTH: $D = \mathbb{R} \setminus \{0\}$

graphs have points $(0, 0), (1, 1)$
(right end) as $x \rightarrow \infty, y \rightarrow \infty$

ODD: $R = \mathbb{R} \setminus \{0\}$

EVEN: $R = \mathbb{R} \setminus \{y \leq 0\}$

$\bullet y = -\frac{2}{3}x^{\frac{4}{7}}$ even or odd symmetry
 $y = \frac{2}{3}\sqrt[7]{x^4}$
 ↑
 variable has even power; even

(1b) Polynomials and Monomials

Monomial → a product of real numbers and variables where the variables have exponents that are non-negative integers and non-fractional. (exponents can only be whole numbers)

are these monomials?

• $7x^2y^3z^{-1}$

NO!
no negative exponents!

• $-\frac{1}{5}y^7a^8$

YES!

• x^2ty^2

NO!

addition separates!

monomial = ONLY multiplication

Are those

monomials?

Polynomial → the finite sum of monomials

• $\frac{1}{5}x^2 + 3^{-1}x^3 + 5y^7$

YES YES YES

• $5y^2 - 3x^{-1} + 8xy^{-7}$

YES NO NO

(#s can
have Θ
exponents)

① Look at each term.
If each term is a monomial, then it is a polynomial.

variables can't have Θ exponents

∴ NO!

∴ YES!

Degree and Leading Coefficient of a polynomial

Degree of a monomial → the sum of the exponents on the variables

Degree of a polynomial → the monomial with the largest degree becomes the degree of the polynomial.

Find degree

→ • $5x^4y^4 + 7x^3y^1 + 8x^2y^7$

1+4 3+1 2+7

= 5 = 4 = 9

① Find degree of each monomial

② Choose largest degree

∴ degree of polynomial is 9.

Leading coefficient → the # that is being multiplied with the monomial that has the highest degree.

∴ leading coefficient of polynomial is 8.

Find degree + leading coefficient for each.

• $y = 3x - 9x^3 + 10x^6 - x^2 + 8$

$$y = 10x^6 - 9x^3 - x^2 + 3x + 8$$

6 3 2 1

∴ degree of polynomial = 6

∴ leading coefficient = 10

① Write in descending order to help identify degree of polynomial.

② Find degree of each monomial.

(In this case, just look for highest exponent)

③ To find leading

coefficient, state coefficient of term with highest (monomial) degree.

• $y = -2(x-4)(3x^3 - x^2)(1-x^4)$

① FOIL all binomial terms and simplify into one expression with ONLY monomial terms.

$$y = 6x^8 - 26x^7 + 6x^6 - 6x^4 + 26x^3 - 8x^2$$

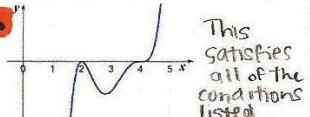
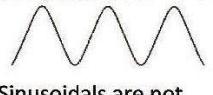
∴ degree of polynomial = 8

② Find the degree of all monomial terms (or in this case, look for highest exponent) to determine degree of the polynomial.

③ Look at coefficient of monomial with highest degree to find the leading coefficient.

∴ leading coefficient = 6

④ Determine if something is a polynomial or not.

	Graphs	Equations	Tables																																			
How to determine:	- smooth (no sharp points) - continuous (domain is all real numbers, no holes, no asymptotes, not piecewise) - curves have infinite end behaviour - no vertical slope	- all terms are of the form ax^n where $a \in \mathbb{R}$ $\{n \in \mathbb{N}\}$ ($a \neq 0$, all terms are monomials)	- the y-values have the same constant difference somewhere (can be 1 st , 2 nd , 3 rd , ...) (always check to see if x's are constant!)																																			
Example	 This satisfies all of the conditions listed above!	$y = \sqrt{3}x^2 - 7^{-1}x^4 + 6x$ (All variables have whole number exponents.)	<table border="1"> <tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr> <tr><td>y</td><td>3</td><td>5</td><td>19</td><td>57</td><td>131</td><td>253</td></tr> <tr><td>Δy</td><td>2</td><td>14</td><td>38</td><td>74</td><td>122</td><td></td></tr> <tr><td>$\Delta \Delta y$</td><td>12</td><td>24</td><td>36</td><td>48</td><td></td><td></td></tr> <tr><td>$\Delta \Delta \Delta y$</td><td>12</td><td>12</td><td>12</td><td></td><td></td><td></td></tr> </table>	x	0	1	2	3	4	5	y	3	5	19	57	131	253	Δy	2	14	38	74	122		$\Delta \Delta y$	12	24	36	48			$\Delta \Delta \Delta y$	12	12	12			
x	0	1	2	3	4	5																																
y	3	5	19	57	131	253																																
Δy	2	14	38	74	122																																	
$\Delta \Delta y$	12	24	36	48																																		
$\Delta \Delta \Delta y$	12	12	12																																			
Non-Example		$y = \frac{5x^4 - x^3}{x}$ After simplifying, x in the denominator produces the restriction $x \neq 0$, which is a hole.	<table border="1"> <tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr> <tr><td>y</td><td>4</td><td>5</td><td>7</td><td>11</td><td>19</td><td>35</td></tr> <tr><td>Δy</td><td>1</td><td>2</td><td>4</td><td>8</td><td>16</td><td></td></tr> <tr><td>$\Delta \Delta y$</td><td>1</td><td>2</td><td>4</td><td>8</td><td></td><td></td></tr> <tr><td>$\Delta \Delta \Delta y$</td><td>1</td><td>2</td><td>4</td><td></td><td></td><td></td></tr> </table> <p>If differences are repeating, it is exponential. ∴ not poly.</p>	x	0	1	2	3	4	5	y	4	5	7	11	19	35	Δy	1	2	4	8	16		$\Delta \Delta y$	1	2	4	8			$\Delta \Delta \Delta y$	1	2	4			
x	0	1	2	3	4	5																																
y	4	5	7	11	19	35																																
Δy	1	2	4	8	16																																	
$\Delta \Delta y$	1	2	4	8																																		
$\Delta \Delta \Delta y$	1	2	4																																			

(D)b)

→ You can use this method to find y-intercept.
Just apply all powers to numerical terms in brackets & take single numbers into account!

How to find degree/leading coefficient for polynomial in factored form (shortcut)

• $y = -2(x-4)(3x^3-x^2)(1-x^4)$

ROUGH:

$$\begin{aligned} & (x)(3x^3)(-x^4)(-2) \\ & = 3x^4(-x^4)(-2) \\ & = -3x^8(-2) \\ & = 6x^8 \end{aligned}$$

$\underbrace{\hspace{1cm}}$
Leading term

② FOIL these terms to find your leading coefficient/degree.

- In each bracket, look at all the x's and pull them out (with negatives/coefficients if applicable). Also pull out any exponents they must be raised to. * Pull out greatest exponents.

* Take into account any single numbers!

• $y = -x^3(4-2x)(3x+5)^2$

(Follow steps from previous example)

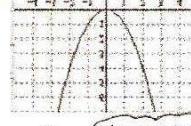
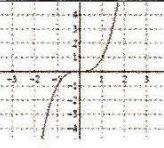
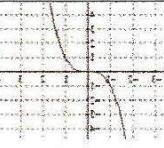
ROUGH: $(-x^3)(-2x)(3x)^2$

$$\begin{aligned} & = (2x^4)(9x^2) \\ & = 18x^6 \end{aligned}$$

$\underbrace{\hspace{1cm}}$
Leading term

∴ degree=6 leading coefficient=18

②a) Properties/graphs of polynomials

	Domain & Range	End Behaviour if (+) Leading Coefficient	End Behaviour if (-) Leading Coefficient
Even Degree Polynomials	Domain: $\{x \in \mathbb{R}\}$ Range: $\{y \in \mathbb{R} y \geq 0\}$ OR $\{y \in \mathbb{R} y \leq 0\}$ <small>note: zero can change and become min. value</small>	 $\lim_{x \rightarrow \pm\infty} y = \infty$ (in this case) $\lim_{x \rightarrow \pm\infty} y = \infty$	 $\lim_{x \rightarrow \infty} y = -\infty$ $\lim_{x \rightarrow -\infty} y = -\infty$ (in this case) $\lim_{x \rightarrow \pm\infty} y = -\infty$
Odd Degree Polynomials	Domain: $\{x \in \mathbb{R}\}$ Range: $\{y \in \mathbb{R}\}$	 $\lim_{x \rightarrow \infty} y = \infty$ $\lim_{x \rightarrow -\infty} y = -\infty$	 $\lim_{x \rightarrow \infty} y = -\infty$ $\lim_{x \rightarrow -\infty} y = \infty$

States which quadrants end behaviour occurs in

How can end behaviour be described?

Informally: "up, up"

(as x goes up, y goes up)

"quadrant II to quadrant I"

Formally: "as $x \rightarrow \pm\infty$, $y \rightarrow \pm\infty$ "

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$$

(2b) Number of Roots and number of extreme values (max/min or turning points)

	Number of Zeros	Number of turning points
Even Degree Polynomials	<p>Degree n has at most n number of zeros. The lowest number of zeros possible is none. (The parabola-like function could be shifted above x-axis and approach infinity so that it never touches the x-axis.)</p> <p>$\bullet y = -4x^2 + x^4 + x + 1$ leading term = x^4 degree = 4 \therefore # of possible zeros = 0, 1, 2, 3, 4</p>	<p>Degree n has at most $(n-1)$ turning points. Possible #s can be found by skip counting odd numbers.</p> <p>Lowest # of turning points: 1 Highest # of turning points: $(n-1)$</p> <p>$\bullet y = -4x^2 + x^4 + x + 1$ leading term = x^4 degree = 4 max turning points = $(n-1) = (4-1) = 3$ \therefore # of possible t.p = 3, 1</p>
Odd Degree Polynomials	<p>Degree n has at most n number of zeros. The lowest number of zeros possible is 1. (All cubic-like functions must touch x-axis at least once because of its range.)</p> <p>$\bullet y = 0.25x^5 - x$ leading term = $0.25x^5$ degree = 5 \therefore # of possible zeros = 1, 2, 3, 4, 5</p>	<p>Degree n has at most $(n-1)$ turning points. Possible #s can be found by skip counting even numbers.</p> <p>Lowest # of turning points: 0 Highest # of turning points: $(n-1)$</p> <p>$\bullet y = 0.25x^5 - x$ leading term = x^5 degree = 5 max turning points = $(n-1) = (5-1) = 4$ \therefore # of possible t.p = 4, 2, 0</p>

To determine turning points: find max # of turning points using $(n-1)$ then skip count down using even/odd numbers until you reach lowest possible # of turning points for the polynomial. (Lowest # is based on if polynomial is odd or even.)

(2c) Polynomials in factored form \rightarrow sketching

* to find degree/leading coefficient of polynomials in factored form, refer to (1b) shortcut *

$$\bullet y = x^5(-2x-10)^4(4x+9)$$

$$x=0$$

$$-2x-10=0$$

$$x = -5$$

$$4x+9=0$$

$$x = -\frac{9}{4}$$

② Draw an x-axis ONLY

and plot your zeros.

① Find all the zeros of this polynomial by making each bracket equal to zero (as you would for a quadratic)

$$\text{Polynomial: } (x^5)(-2x)^4(4x)$$

$$= (x^5)(16x^4)(4x)$$

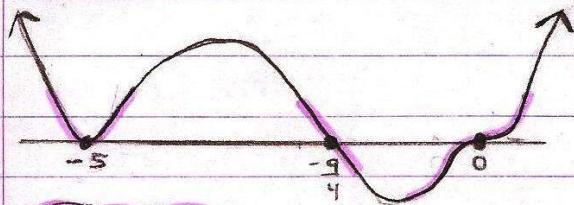
$$= 64x^{10}$$

degree = 10 = even!

leading coefficient = 64 = positive!

using these:
we know polynomial
graph will look like
...↑

- ③ Determine the end behaviour of this polynomial using degree + leading coefficient



- ④ Draw the end behavior of the graph.
★ This helps you start your graph.

If powers on zeros are:

- = 1 → think "cut" and draw linear piece going through zero.
- even and > 1 → think "bounce" and draw a quadratic-like piece going through zero.
- odd and > 1 → think "bend" and draw a cubic-like piece going through zero.

$x = -5 \rightarrow$ power of 4 → bounce

$x = -\frac{9}{4} \rightarrow$ power of 1 → cut

$x = 0 \rightarrow$ power of 5 → bend

★ When sketching, don't worry about y-values or how high/low to make your graph.

- ⑤ Look at the powers corresponding to each of the zeros to decide how to draw each.

★ When sketching,
start from an end behaviour piece and draw bounce/cut/bend according to direction you are going.

$$b^2 - 4ac < 0$$

$$(2a)^2 - 4a(a-2) < 0$$

$$4a^2 - 8a < 0$$

$$4a(a-2) < 0$$

$$a=0 \quad a=2$$

$$\begin{array}{c} 0 \\ \hline + \quad - \quad + \\ \times \quad \checkmark \quad \times \end{array}$$

$$(0, 2)$$

Sub-in the values that we have for b and c.

common factor and then test for intervals to solve for a.

- ⑥ Write your \therefore statement with your entire quadratic in terms of a.

$$\therefore y = (x+1)(x+5)a (ax^2 + 6ax + 5a)$$

THIS IS THE FAMILY OF EQUATIONS FOR THIS POLYNOMIAL

③(c) How to find a quadratic/cubic polynomial equation

* Refer to journal #1b) in Unit 3 on how to use a chart of differences to find the unknown variables for your equation.

③(a) Transformations of Polynomials

Sketching

$$y = -2(3(x-4))^5 - 1$$

$$\text{parent } x^5$$

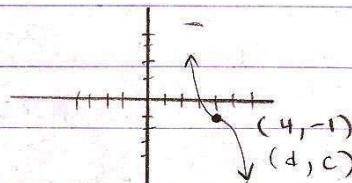
a = -2 = vertical stretch, reflect in x-axis

K = 3 = horizontal compression

d = 4 = shift 4 units right

c = -1 = shift 1 unit down

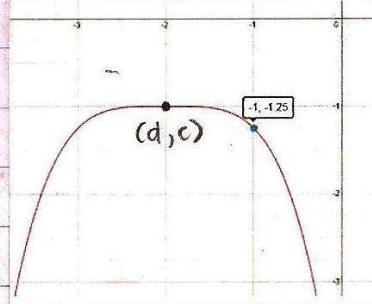
① State all transformations of the parent graph
(To help, identify the parent graph)



- ② Draw a coordinate grid and plot your saddle point.
 $y = x^5$ has one at (0,0) so your new one will be (d, e)
Then apply any reflections.
Stretch/compression do not need to be shown.

When to use transformed form: when graph has no wobbles or if given a, k, d, c (or at least some values and a point)

Finding an equation of a polynomial \rightarrow transformed form



- ③ List transformations to $y = x^4$

$a = ? \Rightarrow$ reflected in x -axis

$k = 1$ (let $k = 1$ so that you only find your a -value by using a point + sub-ing in.)

$d = 2 \Rightarrow$ shift left 2 units

$c = -1 \Rightarrow$ shift down 1 unit

- ⑤ Use point $(-1, -1.25)$ to find a -value

$$-1.25 = a(-1+2)^4 - 1$$

$$-1.25 = a(1)^4 - 1$$

$$-1.25 = a - 1$$

$$\boxed{-0.25 = a}$$

- ① Decide if you will have to use an odd/even power function.

Both ends of graph are approaching $-\infty$, therefore this power is even.

- ② Think about the lowest-possible (but most-correct) degree to use.

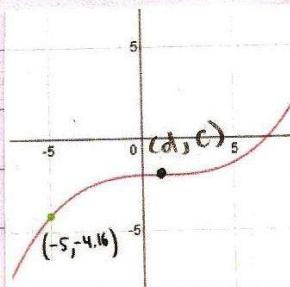
bottom of graph is too flat to be a quadratic. So, use a quartic (degree 4) as your parent function.

- ④ Plug known values into transformed form equation.

$$y = a(x+2)^4 - 1$$

- ⑥ Write your completed equation

$$\boxed{\therefore y = -0.25(x+2)^4 - 1}$$



- ① Decide to use odd/even power function.

- ② Think about which degree to use. (Think lowest, but still, most-correct)

parent: $y = x^3$

- ③ List transformations to $y = x^3$

$a = ? \quad k = 1 \quad d = -1 \Rightarrow$ shift right 1 unit

$c = -2 \Rightarrow$ shift down 2 units

- ④ Plug in known values into transformed form equation.

$$y = a(x+1)^3 - 2$$

- ⑤ Sub in point $(-5, -4.16)$ to find a -value

$$-4.16 = a(-5+1)^3 - 2$$

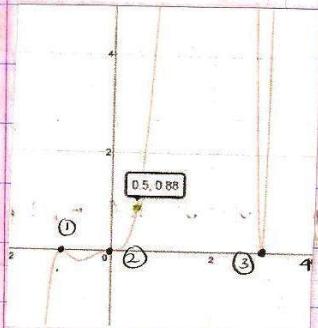
$$-4.16 = -216a - 2$$

$$\boxed{0.01 = a}$$

$$\boxed{\therefore y = 0.01(x+1)^3 - 2}$$

when to use factored form: when graph has wobbles or when given roots + their corresponding multiplicities (and a point to solve for a/k)

Finding an equation of a polynomial \rightarrow factored form



② List all the roots visible on the graph

$$(x+1)(x)(x-3)$$

① ② ③

③ Look at how the graph touches each of these roots to determine the multiplicity of each root. (look for cut, bend, bounce)

$(x+1) \rightarrow$ bounce \rightarrow power 2

$(x) \rightarrow$ bend \rightarrow power 3

$(x-3) \rightarrow$ bounce \rightarrow power 2

④ Determine the overall degree of the polynomial by looking at the end behaviour.

overall degree = odd

$(\lim_{x \rightarrow -\infty} y = -\infty \text{ and } \lim_{x \rightarrow \infty} y = \infty)$

④ Write everything you know about the polynomial and write in an a for the a-value that is TBD.
 $y = a(x+1)^2(x)^3(x-3)^2$

check: overall degree = odd

$$2+3+2 = 7 = \text{odd}$$

⑤ Using point $(0.5, 0.88)$ find the a-value

Find the a-value

$$0.88 = a(0.5+1)^2(0.5)^3(0.5-3)^2$$

$$0.88 = a(1.5)^2(0.5)^3(-2.5)^2$$

$$0.88 = a(2.25)(0.125)(-6.25)$$

$$\underline{0.88 = 1.7578125a}$$

~~$$1.7578125$$~~

$$0.5006 \div a$$

$$\frac{2816}{5625} = a$$

⑥ Write your complete equation

$$\therefore y = \frac{2816}{5625} (x+1)^2(x)^3(x-3)^2$$

Dividing Polynomials

④a) Basics

Divisor → what something is being divided by

Dividend → what is being divided

Quotient → how many full/complete times something can be divided into something else

Remainder → leftover portion that cannot be fully/completely divided by something.

"MATHEMATICALLY" speaking...

A divisor is a quantity used to divide a dividend.

A dividend is a quantity being divided by a divisor.

A quotient is an amount expressing the number of complete times the dividend can be divided by the divisor.

The remainder is an amount expressing a quantity that could not be divided by the divisor to produce a "complete divisor".

Statement Formulas

$$\textcircled{1} \quad \frac{\text{Dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}$$

$$\textcircled{2} \quad \text{dividend} = (\text{quotient})(\text{divisor}) + \text{remainder}$$

Why do these make sense? (Note, second one just got rid of denominator of divisor by multiply both LS and RS by the divisor.)

$$28 \div 3 = ?$$

28 → dividend

3 → divisor

think → $\frac{28}{3}$ = What as a mixed number? (that becomes our answer)

$$\frac{28}{3} = 9 \frac{1}{3} \quad \begin{array}{l} \rightarrow \text{quotient} = 9 \\ \rightarrow \text{remainder} = 1 \end{array} \quad (\text{and } 3 \text{ is still divisor})$$

$$\therefore \frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}$$

$$\therefore \frac{28}{3} = 9 + \frac{1}{3}$$

(answer recorded
as needed for this
unit.)

$$-2x^4 + 12x^3 - 3x + 4 = (-2x^3 - 2x + 16)(x^3 - x + 3) + 19x - 44$$

How to divide using long division

• $\frac{-2x^4 + 12x^3 - 3x + 4}{x^3 - x + 3}$ ← dividend

← divisor

quotient
divisor) dividend

$$\begin{array}{r} -2x^2 - 2x + 16 \\ x^3 - x + 3) -2x^4 + 0x^3 + 12x^3 - 3x + 4 \\ - -2x^4 + 2x^3 - 6x^2 \\ \hline -2x^3 + 18x^2 - 3x \\ - -2x^3 + 2x^2 - 6x \\ \hline 16x^2 + 3x + 4 \\ - 16x^2 - 16x + 48 \\ \hline 19x - 44 \end{array}$$

REMAINDER

- ⑥ After carrying down that term, look at THE FIRST TERM of the expression and how many times THE FIRST TERM of the divisor can go into it. Write your answer above.

- ⑦ (Refer to steps 3-5 to complete question)

- ⑧ After you have carried down all terms, the difference you are left with is your remainder.

- ⑨ Write your final statement: -

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\frac{-2x^4 + 12x^3 - 3x + 4}{x^3 - x + 3} = -2x^2 - 2x + 16 + \frac{19x - 44}{x^3 - x + 3}$$

↑ Recording it like this will help for rationals unit
other way is needed for polynomials unit (this)

① Identify the dividend + divisor to make sure you correctly place them before dividing.

② When writing the dividend, make sure to write beginning with the highest power and descend to lowest (zero power). If some powers are "missing" insert a placeholder.

③ To begin dividing, look at how many times THE FIRST TERM of the divisor can go into THE FIRST TERM of the dividend. Write your answer above.

(hint: $\frac{-2x^4}{x^2} = -2x^2$)

④ Multiply the answer you just wrote by THE ENTIRE DIVISOR and write it below the dividend. Then, subtract the expression with the first three terms of the dividend.

- ⑤ After that has been done, carry down the fourth term of the dividend.

*this only works with linear divisors!

How to divide using synthetic division

• $6x^3 - 11x^2 - 7$

$\overline{3x-4}$

$$\begin{array}{c|cccc} \frac{4}{3} & 6 & -11 & 0 & -7 \\ \downarrow & 8 & -4 & -\frac{16}{3} \\ 6 & -3 & -4 & \hline -37 & 3 \end{array}$$

← remainder

① Set up your "synthetic division box". In your box, write down the coefficients of the dividend (MAKE SURE VARIABLES ARE WRITTEN IN DESCENDING ORDER + use placeholders when necessary).

- If $\text{denom} \neq 0 : 3x-4 \neq 0$

$$\begin{array}{r} +4 \quad +4 \\ \hline \frac{3x-4}{3} \quad \frac{3}{3} \end{array}$$

$$\left| \begin{array}{c} x \neq \frac{4}{3} \end{array} \right.$$

put $\frac{4}{3}$ to left

of the division box.

② Confirm that your divisor is linear. To determine the number to go to the left of your "synthetic division box", think of what x -value would make the divisor equal to zero. (AKA, what would you restrict x to be so that $\text{denom} \neq 0$)

To begin dividing...

③ Carry down first number in your box.

④ Multiply your $\frac{4}{3}$ (restricted x -value) by the number you just carried down. Write the product underneath the second number. Then, add.

⑤ Repeat step 4 until you reach the last column.

The number that you get as a sum in your last column is YOUR remainder.

The numbers that you get to the left of your remainder are coefficients of your quotient. The exponent of each variable for each term starts at zero (directly left of remainder) and increases to the left.

⑥ Write your final statement.

*NOTE: in this case, divisor wasn't $(3x-4)$, but $(x-\frac{4}{3})$

dividend = (quotient)(divisor) + quotient

$$6x^3 - 11x^2 - 7 = (6x^2 - 3x - 4)\left(x - \frac{4}{3}\right) - \frac{37}{3}$$

If given a divisor that is linear with a coefficient in front of the x , the number that you use to the left of your box in synthetic division becomes your new divisor. So, record your final answer using that.

Then, if your quotient has a common factor that you can use to "recreate" your old divisor.

- If our quotient for the question in the example on the previous page was $(6x^2 - 3x + 9)$, look at how we could factor and "recreate" the original divisor.

$$(6x^2 - 3x + 9)(x - \frac{4}{3})$$

↑
FACTOR 3

$$3(2x^2 - x + 3)(x - \frac{4}{3}) = (2x^2 - x + 3)(3x - 4)$$

↓
multiplying by 3

↑
"new" quotient
↑
original divisor

④(d) IMPORTANT NOTES

→ You must add zeros as placeholders when a divisor/dividend's variables have "missing" powers. When variables have "missing" powers, it just means that when terms of a polynomial are written in descending order, the powers on the variables should decrease by 1. If not, insert a zero coefficient and a variable with the "missing" power.

A zero must also be used as a placeholder in synthetic division if necessary!

→ Synthetic division only works with linear divisors.

→ When linear divisors have a coefficient on the x , your final answer must be manipulated before writing your final statement. See note above + example on previous page.

Theorems

- ⑤ a) Remainder Theorem + how to use it.

Remainder Theorem: When polynomial $f(x)$ is divided by $(x-a)$, then $f(a) = \text{remainder}$.

(proof) let divisor = $(x-a)$

let dividend = $p(x)$

dividend = (quotient)(divisor) + remainder

$p(x) = (\text{quotient})(x-a) + \text{remainder}$

Sub-in $x=a$ (Do: $f(a)$)

$f(a) = (\text{quotient})(a-a) + \text{remainder}$

$f(a) = (\text{quotient})(0) + \text{remainder}$

$f(a) = \text{remainder}$

- * Use the remainder theorem to find the value of k when

$p(x) = 2x^3 + 8x^2 - kx + 1$ is divided by $(x+2)$ and the remainder is 27.

$(x+2)$ according to theorem: $p(a) = \text{rem.}$ ① Think about how you
 \uparrow can use the remainder
 $a = -2$ $p(-2) = 27$ Theorem.

\therefore The entire RS must be equal to 27 when the input of -2 is used.
 If I input -2 into every x on the RS, I will have one
 equation equal to 27 with one unknown k to solve.

$$p(-2) = 2(-2)^3 + 8(-2)^2 - k(-2) + 1$$

② Plug in -2 for x .

$$27 = 2(-8) + 8(4) + 2k + 1$$

③ Drop $p(-2)$ and make

$$27 = -16 + 32 + 2k + 1$$

RS = 27. Then evaluate
 and solve for k .

$$+16 +16 -32 -1$$

$$\begin{array}{r} -32 \\ -1 \\ \hline 10 \end{array}$$

$$\frac{10}{2} = \underline{\underline{k=5}}$$

- * What is the remainder when $p(x) = 12x^3 - 11x^2 + 17x - 6$ is divided by $(3x-2)$?
 with divisor $(x-a)$, $p(a) = \text{remainder}$

④ Identify remainder

$$3x-2=0$$

$$p\left(\frac{2}{3}\right) = 12\left(\frac{2}{3}\right)^3 - 11\left(\frac{2}{3}\right)^2 + 17\left(\frac{2}{3}\right) - 6$$

theorem. Then find
 your necessary inputs.

$$3x=2$$

$$\left| \begin{array}{l} x=\frac{2}{3} \\ \downarrow a \end{array} \right.$$

$$p\left(\frac{2}{3}\right) = 12\left(\frac{8}{27}\right) - 11\left(\frac{4}{9}\right) + 17\left(\frac{2}{3}\right) - 6$$

② After you have found a ,
 input + evaluate to find rem.

$$\boxed{p\left(\frac{2}{3}\right) = 4}$$

\therefore The remainder of polynomial $P(x)$ is 4.

⑤b) Rational Root Theorem + how to use it

Rational Root Theorem: the polynomial in the form of

$f(x) = a_n x^n + \dots + a_1 x^1 + a_0$ has possible roots in the form of $\pm \frac{p}{q}$

→ where p is defined by factors of a_0 (last term) and where q is defined by factors of a_n (leading coefficient)

NOTE: coefficients must be integers to use this theorem.

If coefficients aren't integers, find LCD, etc. and create a new equation. This new equation will have the same roots because it is part of the same family of equations.

* The rational root theorem is used to list possible rational zeros. *

- $f(x) = x^3 - \frac{1}{2}x^2 + \frac{1}{3}x - 2$ LCD = 6

$$\times 6 \quad x 6 \quad x 6 \quad x 6$$

$$g(x) = 6x^3 - 3x^2 + 2x - 12$$

$$\underbrace{6}_{q} \quad \underbrace{-3}_{p}$$

$$\frac{\pm p}{q} : \frac{\pm 12}{6}, \frac{\pm 6}{6}, \frac{\pm 4}{6}, \frac{\pm 3}{6}, \frac{\pm 2}{6}, \frac{\pm 1}{6}$$

possible rational roots $\Rightarrow \pm 2, \pm 4, \pm 6, \pm 12,$
 (for $f(x)$ and $g(x)$) $\pm 1, \pm 3, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{3}{2},$
 $\pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}$

① NOTE: coefficients are not all integers. So, find LCD and create a new equation. Don't forget to rename it!

② Use the rational roots theorem on $g(x)$ instead of $f(x)$.
 * ROOTS WILL BE SAME

③ Identify p + identify q
 Then write all factors for each. (Consider \pm)

④ Create individual fractions to express every \pm possible rational root

⑤c) Factor Theorem + how to use it to factor a cubic/quartic Fully

Factor Theorem: The divisor $(x-a)$ is a factor of a polynomial $f(x)$ if $f(a)=0$.

• Use the factor theorem to factor $f(x) = 2x^3 - 9x^2 + x + 12$ with a factor $(x-4)$.

$(x-a)$ if $f(a)=0$

$(x-4)$ if $f(4)=0$ ✓

① Identify the factor theorem.

TO FULLY FACTOR A CUBIC POLYNOMIAL, WE NEED THREE FACTORS:

$$(x-4)(ax^2 + bx + c) \quad \text{OR} \quad (x-4)(x+r)(x+s)$$

Example of how to factor using synthetic division

GENERAL STEPS

- ① Start with the rational root theorem to find a single factor.
(Make a list of possible factors)

before this
you can
use
Descartes's
rule of
signs

- ② Use the factor theorem to test zeros and find 1 factor.
- ③ Use this factor as a divisor and divide the polynomial by it.
- ④ Factor the quotient fully

$$\bullet 2x^3 - 9x^2 + 10x - 3 = f(x)$$

$$\frac{+P}{2} : \frac{\pm 3, \pm 1}{\pm 2, \pm 1} \Rightarrow \frac{\pm \frac{3}{2}, \pm 3, \pm \frac{1}{2}, \pm 1}{2, \pm 1} \text{ Rational Root Th.}$$

$$f(1) = 0 \checkmark \text{ Factor Th.}$$

$\therefore (x-1)$ is a factor

$$\begin{array}{r} 1 | 2 & -9 & 10 & -3 \\ & \downarrow & 2 & -7 & 3 \\ & 2 & -7 & 3 & | 0 \end{array}$$

$$\therefore \text{quotient} = 2x^2 - 7x + 3$$

$\therefore \text{FULLY FACTORED} =$

$$(x-1)(2x^2 - 7x + 3)$$

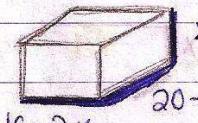
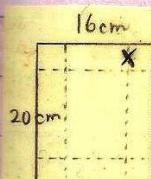
$$(2x^2 - 7x + 3) \quad \frac{1}{2} \times \frac{1}{3}$$

$$= (2x-1)(x-3)$$

Solving Word Problems

6(a) Volume of a box example

- A 16cm x 20cm sheet of cardboard is to be folded as per the diagram to make an open top box with a volume of 384 cm³. Find its possible dimensions.



- ② Identify each measurement as l, w, h

$$l = 16 - 2x$$

$$w = 20 - 2x$$

$$h = x$$

$$V = 384 \text{ cm}^3$$

$$V = lwh$$

① Label the uniform length of the squares that are being cut as x.
This becomes your height and affects your length/width.
Draw a 3D box and label new length/width.

Volume of a rectangular prism.

$$V = lwh$$

$$384 = (16-2x)(20-2x)(x)$$

At this point, notice that this whole equation is written in terms of one variable, x . So, if you solve for x , you can find the other dimensions.

$$384 = (16-2x)(20-2x)(x)$$

$$384 = (16-2x)(20x - 2x^2)$$

$$384 = 320x - 32x^2 - 40x^2 + 4x^3$$

$$0 = 320x - 32x^2 - 40x^2 + 4x^3 - 384$$

$$f(x) = 4x^3 - 72x^2 + 320x - 384$$

$$f(x) = 4(x^3 - 18x^2 + 80x - 96)$$

Rational Root Th.

$$\pm \frac{P}{Q} : \pm 96, \pm 48, \pm 32, \pm 24, \pm 16, \pm 12,$$

$$\qquad \qquad \qquad \pm 8, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1$$

Descartes's Rule

of poss. + roots: 3 } THINK:
of poss. 0 roots: 0 } This must be
true since
length/width/
height $\neq 0$

Factor Th. $f(2) = 0 \checkmark$

$\therefore (x-2)$ is a factor

Synthetic Div.

$$\begin{array}{r} 1 & -18 & 80 & -96 \\ \downarrow & 2 & -32 & 96 \\ 1 & -16 & 48 & 0 \end{array}$$

$$= x^2 - 16x + 48$$

$$f(x) = (x-2)(x-4)(x-12)$$

$$\boxed{x=2} \quad \boxed{x=4} \quad \boxed{x=12}$$

$$l = 16-2x \quad l = 16-2x \quad l = 16-2x$$

$$w = 20-2x \quad w = 20-2x \quad w = 20-2x$$

$$h = x \quad h = x \quad h = x$$

$$\therefore l = 12 \text{ cm}$$

$$w = 16 \text{ cm}$$

$$h = 2 \text{ cm}$$

$$\therefore l = 8 \text{ cm}$$

$$w = 12 \text{ cm}$$

$$h = 4 \text{ cm}$$

$$\therefore l = -8 \text{ cm}$$

$$w = -4 \text{ cm}$$

$$h = 12 \text{ cm}$$

length/width
are negative. REJECT
this zero / these answers.

③ Plug your V, l, w, h variables into the formula for volume.

④ FOIL and make LS equal to zero. Once you do this, you have changed the volume equation. Name this new one $f(x)$.

⑤ Use the rational root theorem and Descartes's rule of signs to decide which #s to try and see if they are factors.

⑥ Using #s from step 5, use factor theorem to find at least 1 factor.

⑦ Instead of looking for more, use synthetic division from here.

⑧ Factor your quotient and write your factors $f(x)$

⑨ Find the zeros and then find dimensions of the box using each zero.

NOTE: sometimes, even if your zeros are acceptable, once you begin solving for dimensions, you may have to reject some zeros.

(6b) Why is it important to know how to use technology in finding the regression equation for real-life data?

WHAT IT IS... } Regression is a statistical measure that attempts to determine the strength of the relationship between independent/ dependent variables.

Regression factor measures the accuracy of a regression equation to match the data. The closer R^2 is to 1, the more accurate the equation is.

It is important to know how to use technology to formulate regression equations because sometimes, a family of equations with defined constants (ie $a \in (-,)$) isn't enough/ accurate enough to represent a relationship.

Regression technology essentially finds the best values of constants. You would normally define with an interval of numbers.

REMEMBER: real-life data is not exact and will not create a perfect polynomial (chances are LOW).

This means that, for example, differences may be close to being constant but are not equal to. This means that an equation for a polynomial function cannot be derived using differences.

7 Pseudo Quadratic Equations (how to recognize & factor)

• $4x^8 - 61x^4 + 225$ This is a pseudo quadratic because the first variable, x^8 , is squared compared to the second variable, x^4 .

Let $a = x^4$

(because $x^8 = a^2$)

This turns equation (1) Use a substitution. (let $a =$) into a simple quadratic

$4a^2 - 61a + 225$ (2) Factor this quadratic. (Quad formula is quicker with large numbers)

$$a = \frac{61 \pm \sqrt{121}}{2(4)}$$

$$a = \frac{61 \pm 11}{8}$$

$$|a=9\} \text{ or } |a=\frac{25}{4}$$

(3) Find both factors

(4) Insert them into factored form.

$$(a-9)(a-\frac{25}{4})$$

(5) Since we did a Substitution, we must do a re-substitution.

When to stop factoring: when all variables are power 1.

$$\begin{aligned} & (a-9)(a-\frac{25}{4}) \quad (x^2-\frac{\sqrt{25}}{4})(x^2+\frac{\sqrt{25}}{4}) \\ & (x^4-9)(x^4-\frac{25}{4}) \quad = (x^2-\frac{5}{2})(x^2+\frac{5}{2}) \\ & \text{diff of sq} \quad (x^2-3)(x^2+3)(x^2-\frac{5}{2})(x^2+\frac{5}{2}) \\ & \text{diff of sq} \end{aligned}$$

⑥ Factor from here until FULLY factored.

$$\therefore (x+\sqrt{3})(x-\sqrt{3})(x^2+3)(x+\sqrt{\frac{5}{2}})(x-\sqrt{\frac{5}{2}})(x^2+\frac{5}{2})(4)$$

⑦ After fully factoring, compare all variables multiplied together with the leading coefficient of the original polynomial. Add in any coefficients if needed.

$$\Rightarrow (x)(x)(x^2)(x)(x)(x^2) = x^8 \quad \left. \begin{array}{l} \text{Leading Coeff.} = 4x^8 \\ \therefore \text{odd factor of 4 because} \\ \text{fully factored answer is missing this.} \end{array} \right\}$$

This example identifies a different pseudo quadratic pattern.

$$x^9 - x^6 - x^3 + 1$$

$$\text{let } a = x^3$$

① Use substitution. Create a let statement.

$$\text{because: } a = x^3$$

$$a^2 = (x^3)^2 = x^6$$

$$a^3 = (x^3)^3 = x^9$$

So, using $a = x^3$, we can create a simple cubic polynomial to factor.

Rational Root Th. $\frac{p}{q}$: ± 1 ② Use rational root theorem and factor theorem to find a factor for this polynomial

factor th., $f(-1) = 0 \checkmark \therefore (a+1)$ is a factor

③ Use synthetic division to find remaining quadratic.

$$\begin{array}{r} | 1 & -1 & -1 & 1 \\ \downarrow & -1 & 2 & -1 \\ 1 & -2 & 1 & 0 \end{array}$$

$$\therefore f(a) = (a+1)(a-1)$$

$$= a^2 - 2a + 1$$

$$= (a-1)(a-1)$$

$$= (a-1)^2$$

④ Since we did a substitution, we must do a re-substitution.

$$\begin{aligned} & (a+1)(a-1)^2 \\ & (x^3+1)(x^3-1)^2 \\ & (x^3+1)(x^3-1)(x^3-1) \end{aligned}$$

sum of cubes difference(s) of cubes

⑤ Factor from here until FULLY factored.

$$(x+1)(x^2-x+1)(x-1)(x^2+x+1)(x-1)(x^2+x+1)$$

IDENTICAL PAIRS OF BINOMIALS

$$\therefore (x+1)(x^2-x+1)(x-1)^2(x^2+x+1)^2$$

⑥ Since there is no coefficient (leading) in the original polynomial, no need to verify if any factor needs to be added.

Example of factoring out smallest exponent

$$\bullet \frac{x^{\frac{1}{3}}(x^2-4)^{\frac{3}{4}}}{x^{\frac{1}{3}}(x^2-4)^{-\frac{1}{4}}} - x^{\frac{7}{3}}(x^2-4)^{-\frac{1}{4}}$$

← write what you are factoring out

$$x^{\frac{1}{3}}(x^2-4)^{-\frac{1}{4}} [(x^2-4) - x^2] \quad \leftarrow \text{CLT}$$

$$x^{\frac{1}{3}}(x^2-4)^{-\frac{1}{4}} (-4) \quad \leftarrow \text{get rid of negative exp.}$$

$$-4x^{\frac{1}{3}}$$

$$(x^2-4)^{-\frac{1}{4}} \quad \leftarrow \text{FACTOR THIS DIFF OF SQ!}$$

NOTE: powers on two binomials must equal $\frac{1}{4}$ when multiplied

$$= \frac{-4x^{\frac{1}{3}}}{(x+2)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}}$$

← FINAL ANSWER