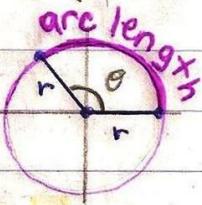


TO REMEMBER: $180^\circ, 90^\circ, 45^\circ, 30^\circ, 60^\circ$

$\pi, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$

④ a) Radians → what are radians?

Radians are a concept used to measure angles using the radius of a triangle (and its circle) and the arc length of an angle on a circle.



In radians: $\theta = \frac{a}{r}$ ← where a is the arc length
↑ and r is the radius.

→ Radians are a value that tells how many radii of a circle can fit along the arc length of the angle θ .

NOTE: RADIANS ARE NOT A UNIT!

How many radians are there in a full circle?

- The arc length of an ENTIRE circle is its circumference.

$$C = 2\pi r$$

If we plug this into our formula for θ (in radians) as our a , let's see what happens.

* it doesn't matter what r is, they cancel.

$$\theta = \frac{a}{r} = \frac{2\pi r}{r} = 2\pi$$

We just found out that θ of an entire circle, is 2π .

This must mean: $360^\circ = 2\pi$

$$180^\circ = \pi$$

↑ This means there are 2π rad in a full circle.

Using these numbers, we can "assign" angles on our cartesian plane a radian measure.

π	$\frac{\pi}{2}$	$\frac{5\pi}{2}$	\dots
180°	90°	450°	\dots
3π	0°	$2\pi, 4\pi, 6\pi, \dots$	
540°	180°	$360^\circ, 720^\circ, \dots$	

$$\left\{ \begin{array}{l} \text{DEG} \rightarrow \text{RAD} \\ \times \frac{\pi}{180} \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{RAD} \rightarrow \text{DEG} \\ \times \frac{180}{\pi} \end{array} \right.$$

(4)b) How to convert degrees to radians

• 80°

$$80^\circ \times \frac{\pi}{180}$$

$$\text{We Know} \rightarrow 180^\circ = \pi$$

equivalent

$$80^\circ \times \frac{\pi}{180}$$

② Multiply your degrees by

$$\frac{\pi}{180^\circ} = 1$$

$$= \frac{80\pi}{180}$$

$$= \frac{4\pi}{9}$$

④ Reduce your coefficients



This fully simplified fraction is equivalent to 80° . How? SEE BELOW

How to convert radians to degrees

• $\frac{4\pi}{9}$

$$\frac{4\pi}{9} \times \frac{180^\circ}{\pi}$$

③ Multiply by

$$\frac{180^\circ}{\pi} = 1$$

$$= \frac{4 \cdot 180^\circ}{9} = \frac{720^\circ}{9}$$

④ Reduce fraction

$$= 80^\circ$$

\leftarrow This angle in degrees is equivalent

to $\frac{4\pi}{9}$. For more info, look at the previous example, we know this is true. SEE ABOVE

① You essentially want to get rid of the degrees unit itself because radians have no unit.

So, you need to multiply by something over a number of degrees.

② You also don't want to change the value of the angle, so you have to multiply by something equivalent to 1.

★ Also, radians involve π to keep fractions low + simplified.

① You essentially want to cancel π and bring back the unit of degrees.

So, you need to multiply by a number of degrees over π .

② What value of degrees over π do we know that is equal to 1?

(Decimal value of radians \rightarrow degrees)

• 1.396

$$1.396 \times \frac{180^\circ}{\pi}$$
$$= 251.28^\circ$$

$$\div 79.98^\circ$$

$$\approx 80^\circ$$

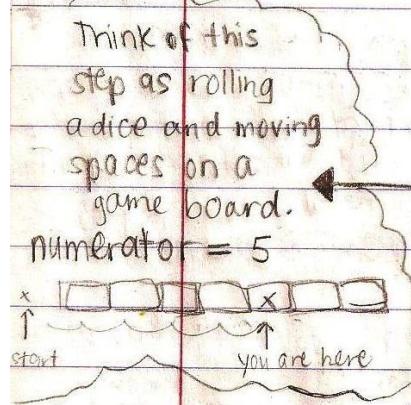
① We know that this is just a rounded version of an angle measured in radians. To convert, multiply by $\frac{180^\circ}{\pi}$

as you would with a normal angle measured in radians.

Notice that 1.396 is 80° because 1.396 is just a decimal approximation of this angle in radians. The exact, as shown on previous page, is $\frac{4\pi}{9} \approx 1.396\dots$

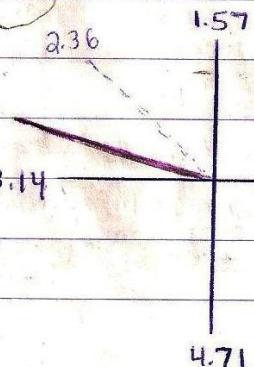
④ c) How to draw radian angles

• $\frac{5\pi}{4}$



★ To determine the accuracy of this sketch, you can convert this number to degrees.

• 2.885



① Think of denominator as # of sections to split 180° (or π) into, on the cartesian plane.

② Draw in points at where you have split the plane. (Think of a clock to do this)

③ Look at your numerator to determine how many times to go around. The spot where you land is where your angle is.

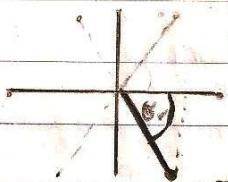
④ Convert quadrant measures to decimal approximations and roughly estimate where you think 2.885 would go. For accuracy, calculate halves b/w each quadrant angle.

* WORKING IN RADIANS

6) a) Find output ratio values given angles (given θ)

• Value of a trig ratio for a special triangle

$$\sin -\frac{\pi}{3}$$



$$Or = \frac{\pi}{3} (= 60^\circ)$$

↑
special angle!

① Draw a picture of the angle using knowledge of how to draw radians

② Find Or to help label sides because triangle may be special.

$$\frac{\pi}{3} (60^\circ) \leftrightarrow \sqrt{3} = y$$

$$r=2 \quad x=1$$

of these values, adjust any negatives before inserting into general trig definition.

$$y = -\sqrt{3} \quad x = 1 \quad r = 2$$

$$\sin -\frac{\pi}{3} = \frac{y}{r} = \frac{-\sqrt{3}}{2}$$

↑
output ratio for $\sin -\frac{\pi}{3}$

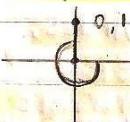
③ Using knowledge of Or , we know Or is across from y .

In a 30-60-90 triangle, we know the other side measures.

④ Use general trig definition for sine to find the output ratio for this angle.

• Value of a trig ratio for a quadrantal angle

$$\sec -\frac{3\pi}{2}$$



$$x=0 \quad y=1 \quad r=1$$

① Draw a picture of the angle.

Note: This is quadrantal

② Since there is no triangle, use unit circle to label its point.

③ Identify x, y, r

④ Use general trig definition for secant to find output ratio of this angle.

$$\sec -\frac{3\pi}{2} = \frac{r}{x} = \frac{1}{0} = \text{undefined}$$

↑
output ratio for
 $\sec -\frac{3\pi}{2}$ is actually,
undefined.

• Value of a trig ratio for neither special nor quadrantal angles

$$\cot(1.35) = [\tan(1.35)]^{-1}$$

\uparrow
This is equivalent to
 $\cot(1.35)$ because it
is $\frac{1}{\tan(1.35)}$ which is the
reciprocal of $\cot(1.35)$.

$$[\tan(1.35)]^{-1} \approx 0.2245 \leftarrow x$$

$\uparrow 1 \quad \leftarrow y$

output ratio for $\cot(1.35)$

① Don't draw this
because you are
not given an exact
angle measure.

This also tells you that
you have to use a
calculator for this ratio.

② There is no "cot"
button on the calculator.
So, use its related
primary trig ratio.

⑥(b) Find input angles given ratios

• If ratios are 0, +1, -1, undefined (quadrantal angles)

$$\csc\theta = \text{undefined} = \frac{r}{y}$$

Now we know $y=0$

$$\theta_1 = 0$$

$$\theta_2 = \pi$$

$$\theta_3 = 2\pi$$

These input
angles satisfy
 $\csc\theta = \text{undefined}$



① For a ratio to be undefined, it must have a denominator of 0.

② Identify the trig definition
for cosecant.

③ Use the unit circle to
determine all the possible
places from 0 to 2π
where $y=0$.

$$\tan\theta = -1$$

$$\tan\theta = -1 = \frac{y}{x}$$

$$y=-1$$

$$x=-1$$

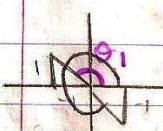
$$y=1$$

*FOR THIS EXAMPLE,
FOLLOW STEPS →

BUT USE APPROPRIATE
VALUES / ADD STEPS
AS NEEDED

④ List the value of the
angles at these locations.

DRAW →



Find θ_r
+ use to find
 θ_1, θ_2

$$\theta_r = 45^\circ = \frac{\pi}{4}$$

45-45-90
triangle

$$\theta_1 = \pi - \frac{\pi}{4} = 180^\circ - 45^\circ = \frac{3\pi}{4}$$

$$\theta_2 = 2\pi - \frac{\pi}{4} = 360^\circ - 45^\circ = \frac{7\pi}{4}$$

These
input
angles
satisfy
 $\tan\theta = -1$

• If ratios have numbers from special triangles

$$\sec \theta = 2 = \frac{r}{x}$$

$$r=2$$

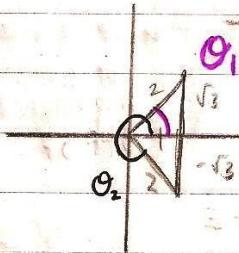
$$x=1$$

$$y=\sqrt{3}$$

$$y=-\sqrt{3}$$

WHEN DRAWING:

think where cosine
is \oplus



① Think about what the definition of secant is to determine if this is special or not.

② Recognize that $r=2$ and $x=1$. These are numbers part of the 30-60-90 triangle. Map out the angles + their corresponding sides by drawing the triangle.

③ Find θ_r to find θ_1 and θ_2

$$\theta_1 = \frac{\pi}{3}$$

$$\theta_2 = 2\pi - \frac{\pi}{3} = 360^\circ - 60^\circ = \frac{5\pi}{3}$$

Both $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ satisfy $\sec \theta = 2$

• If ratios are neither special nor quadrantal

* DON'T NEED TO DRAW TO BEGIN

$$\csc \theta = 3.4219$$

$$\sin \theta = \frac{1}{3.4219}$$

$$\theta = \sin^{-1} \left(\frac{1}{3.4219} \right)$$

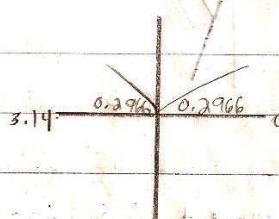
$$\theta_r = 0.2966$$

$$\theta_1 = 0.2966$$

$$\theta_2 = 3.14 - 0.2966 = 180^\circ - \theta_r = 2.8434$$

$$\therefore \theta_1 = 0.2966$$

$$\theta_2 = 2.8434$$



① Find the related acute angle.

② Change ratio to a sine ratio then do sin⁻¹ on this reciprocal ratio.

③ Draw this angle. on the coordinate plane where sine is \oplus

④ Use θ_r to find θ_1 and θ_2

These input angles satisfy $\csc \theta = 3.4219$

YOU
can't do
 \sin^{-1} on
the ratio
unless it
has been
converted
to the
csc reciprocal.

How to sketch a secant graph with transformations

$$y = \sec\left(\frac{\theta}{2} + 135^\circ\right) + 5$$

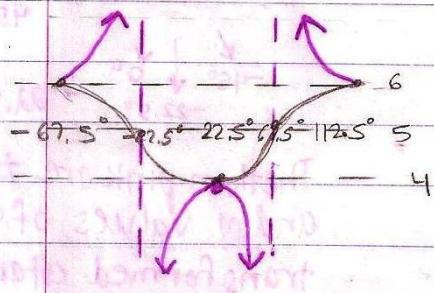
$$y = \cos\left(\frac{1}{2}(\theta + 67.5^\circ)\right) + 5$$

max = 6

axis = 5

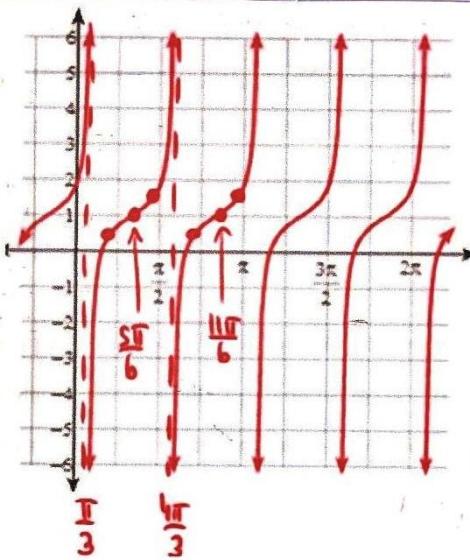
min = 4

$$\text{period} = \frac{360^\circ}{2} = 180^\circ$$



The one full u-shape +
2 half u-shapes and the
two VAs are one cycle of this
transformed secant function/graph.

(2c) Finding an equation of a non-sinusoidal graph



① Identify the trig functions
that could be used to
represent this graph.

- Tangent (positive a-value)
- cotangent (negative a-value)

② Look at the three points
given on the graph. Use
these to find a and c values.

(y-value of middle point) → c-value
(distance between middle
point + highest/lowest) → a-value

$$\begin{cases} c = 1 \\ a = \frac{1}{2} \end{cases}$$

period = (distance between asymptotes)

$$P = \left(\frac{4\pi}{3} - \frac{\pi}{3}\right) = \pi$$

$$P = \pi$$

$$K = \frac{\pi}{\pi} = 1$$

$$K = 1$$

- ③ Identify one period of the graph. Then, use that period to find the k-value.

REMEMBER!

$$K = \frac{\pi}{P} \text{ or } \frac{180^\circ}{P}$$

IMPORTANT: When finding the equation of a graph using two different trig functions, they share the a, k, and c-value! The only thing they don't share is the d-value!!

D-VALUE OF TANGENT:

will always be the x-value of the middle "twisting" point. (of one cycle)

D-VALUE OF COTANGENT:

will always be the x-value/value of the leftmost vertical asymptote. (of one cycle)

- ④ Keeping in mind the note above, find the corresponding d-value of tangent/cotangent.

$$\begin{array}{|c|c|} \hline \text{tangent} & \text{cotangent} \\ \hline d = -\frac{5\pi}{6} & d = -\frac{\pi}{3} \\ \hline \end{array}$$

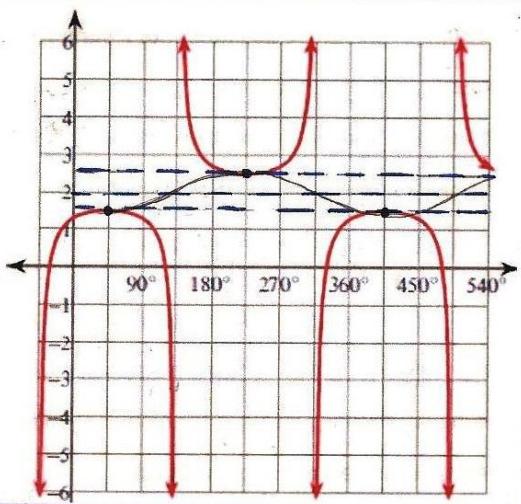
- ⑤ Using the variables that were just found, in steps

1 - 4, create the equations for this graph.

$$\therefore y = \frac{1}{2} \tan\left(x - \frac{5\pi}{6}\right) + 1$$

$$y = -\frac{1}{2} \cot\left(x - \frac{\pi}{3}\right) + 1$$

negative a-value on cotangent because cotangent reflected is the same shape as tangent.



① Identify the trig functions that could be used to represent this graph.

→ Cosecant
→ Secant

② Label the max/min points of the u-shaped graphs by drawing max/min lines on the graph.

Based on the max/min lines just drawn, find the axis.

Then, draw in a sinusoidal wave.

WHY DRAW A SINUSOIDAL WAVE?

We draw a sinusoidal wave because we are looking for equations of sinusoidal reciprocal graphs. If we have the original sinusoidal, we can use it to help us.

By using max/min lines, we know: $c = 2$ $a = \frac{1}{2}$

③ Identify the period of the graph. Then use it to find the K-value.

period = max → max OR min → min

$$P = (405^\circ - 45^\circ) = 360^\circ$$

$$P = 360^\circ$$

$$K = \frac{360^\circ}{360^\circ} = 1$$

$$K = 1$$

Cosecant
(sine wave)

$$d = -135^\circ$$

Secant

(cosine wave) -

$$d = -45^\circ$$

* The cosine wave we referred to when finding d was reflected. SO...

secant has \ominus a-value!

REMEMBER:

$$K = \frac{a\pi}{P} \text{ or } \frac{360^\circ}{P}$$

④ Refer to "important note" in red on previous page.

Find the d-value for the corresponding PRIMARY trig ratio for csc/sec.

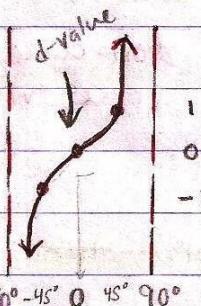
Do this by using the sinusoidal wave drawn on the graph. Compare phase shift of this wave to parent wave.

⑤ Using the variables

found in steps 1-4, create the equations for this graph.

$$\therefore y = \frac{1}{2} \csc(x - 135^\circ) + 2 \quad y = -\frac{1}{2} \sec(x - 45^\circ) + 2$$

③ Parent Graphs of Tangent + Reciprocal Graphs



TANGENT
(one cycle)

period = 180° or π
VA at -90° and 90°
($-\frac{\pi}{2}$ and $\frac{\pi}{2}$)

Domain:

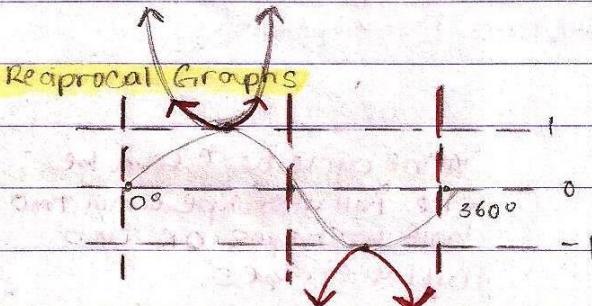
$$\{x \in \mathbb{R} \mid x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}$$

Range: $\{y \in \mathbb{R}\}$

* one cycle here is

one "wave".

Y AND X SWAP IN TANGENT
Y GOES FROM 0 TO 2PI/3



COSECANT

(one cycle)

period = 360° or 2π

VA at every point circ graph crosses HS axis.

VA at $0, \pi, 2\pi$

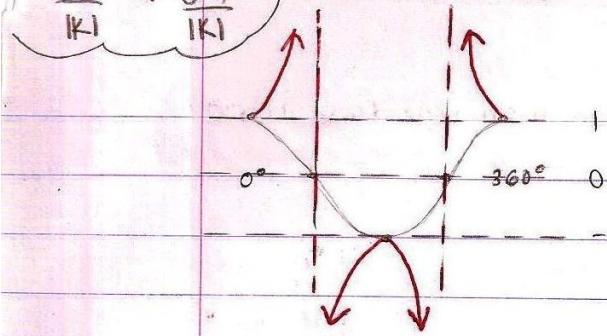
Domain: $\{x \in \mathbb{R} \mid x \neq 0, \pm\pi, \pm 2\pi, \dots\}$

OR $\{x \in \mathbb{R} \mid x \neq \pi n, n \in \mathbb{Z}\}$

Range: $\{y \in \mathbb{R} \mid y \geq 1, y \leq -1\}$

* one cycle here can be
two full U-shapes or
one full U-shape + 2 half
U-shapes.

(to find period for csc/sec or sin/cos:)
 $\frac{360^\circ}{|k|}$ or $\frac{2\pi}{|k|}$



SECANT
(one cycle)

$$\text{period} = 2\pi \text{ or } 360^\circ$$

VA at every point cosine graph touches its axis.

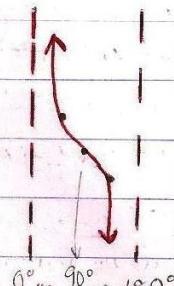
$$\text{VA at } \frac{\pi}{4} \text{ and } \frac{3\pi}{4}$$

$$\text{Domain: } \{x \in \mathbb{R} \mid x \neq \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}, \dots\}$$

$$\text{Range: } \{y \in \mathbb{R} \mid y \geq 1, y \leq -1\}$$

*One cycle here can be one full u-shape and two half u-shapes OR two full u-shapes.

(to find period for tan/cot: 180° or $\frac{\pi}{|k|}$)



COTANGENT
(one cycle)

$$\text{period} = \pi \text{ or } 180^\circ$$

$$\text{VA at } 0^\circ \text{ and } 180^\circ$$

$$- (0 \text{ and } 2\pi)$$

$$\text{Domain: } \{x \in \mathbb{R} \mid x \neq 0, \pm \pi, \pm 2\pi, \dots\}$$

$$\text{Range: } \{y \in \mathbb{R}\}$$

*One cycle here is one "wave"

(3a) How to sketch a tangent graph with transformations

$$y = \frac{1}{2} \tan(2(x - \frac{5\pi}{3})) + 1$$

① Factor to see the k-value/d-value

$$y = \frac{1}{2} \tan(2(x - \frac{5\pi}{6})) + 1$$

② Is a positive or negative?

If negative, draw main 3 points like:

If positive, draw main 3 points like:

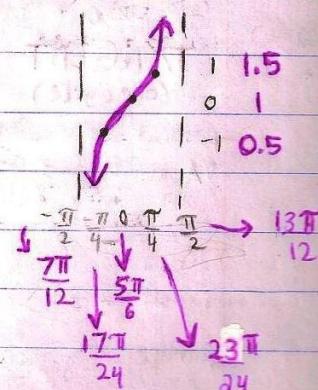
③ Draw in the VAs and label x/y axis using parent graph.

④ Use mapping rule to adjust all the x's and y's (multiply x's by x rule and y's by y rule)

$$(x, y) \rightarrow (\frac{x+d}{k}, ay+c)$$

$$a = \frac{1}{2}, d = \frac{5\pi}{6}, k = 2, c = 1$$

The #s in pink are the x and y values of transformed tangent.



• How to sketch a cotangent graph with transformations:

$$y = \cot(\theta + 90^\circ) + 1$$

① Factor to see k-value/d-value

$$y = \cot(2(\theta + 45^\circ)) + 1$$

② Is a positive or negative?

If positive, draw main 3 points like:

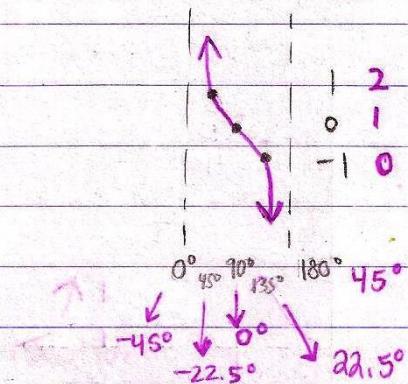
If negative, draw main 3 points like:

③ Draw in the VAs and label x/y axis using parent graph.

④ Using mapping rule, adjust all the x's and y's (multiply x's by x rate. (see multiply y's by y rule)

 mapping rule below)

$$\begin{matrix} a=1 & d=-45^\circ \\ k=2 & c=1 \end{matrix}$$



The numbers in pink are the x and y values of the transformed cotangent.

③ b) How to sketch a cosecant graph with transformations

$$y = -\frac{1}{2} \csc(\theta - 135^\circ) - 2$$

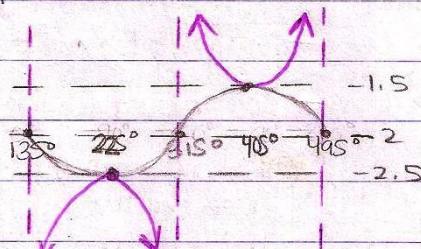
$$y = -\frac{1}{2} \sin(\theta - 135^\circ) - 2$$

$$\text{max} = -1.5$$

$$\text{axis} = -2$$

$$\text{min} = -2.5$$

$$\text{period} = 360^\circ$$



The two U-shaped graphs and the three VAs are one cycle of this transformed cosecant graph/function.

① Rewrite as its corresponding sine/cosine reciprocal.

② Graph the SINE function as normally, you would do.

③ At every point the sine graph crosses the axis, draw a vertical asymptote.

④ At every max point, draw a U-shape going up. At every min point, draw a U-shape going down.