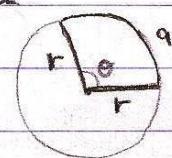


(1a) Arc length → how to find a , θ , r

$$\begin{aligned} a &= \theta r \\ \theta &= \frac{a}{r} \\ r &= \frac{a}{\theta} \end{aligned}$$

*NOTE: θ in all these formula versions is in RADIANS! Why?



What is arclength?
arclength is the distance around the edge of a circle created by angle θ with radius r .

THEORY OF ARCLENGTH

technically, to find your arclength, you would do radius $\times \theta$ (in radians). If you rearrange this and use variables, you will see that your θ can be rewritten as $\frac{\text{arclength}}{\text{radius}}$ because your θ in radians will be a proportion equal to this.

- A space shuttle 200 miles above the earth is orbiting the earth once every 6 hours. How long, in hours, does it take for the space shuttle to travel 8,400 miles? (The distance of the space shuttle from the earth's centre is 4200 miles)

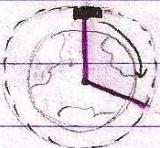
① Identify what you know/
what you need to find.

$$r = 4200 \text{ mi}$$

$a = 8400 \text{ mi}$ ← we are given the distance it has orbited along the dotted line
 $\theta = ?$

② Use $\theta = \frac{a}{r}$ to solve for θ

$$\theta = \frac{8400 \text{ mi}}{4200 \text{ mi}} = 2 \quad \therefore \theta = 2 \text{ (in radians!)}$$



position after 1 hour

③ How can you use this measure of θ to find time?

→ use the frequency where 1 revolution occurs every 6 hours.

create a ratio to compare.

$$\frac{\theta \text{ of revolutions(s)}}{\text{hour(s)}} : \frac{\theta \text{ of revolutions(s)}}{\text{hour(s)}}$$

$$\frac{2\pi}{6 \text{ hours}} : \frac{2}{t}$$

use a related equation to solve.

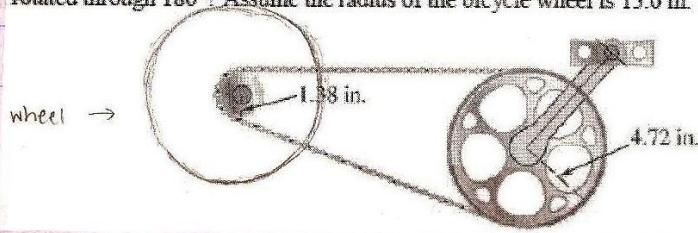
$$\frac{2\pi}{6 \text{ hours}} \times \frac{2}{t} = \frac{12 \text{ hours}}{2\pi}$$

$t = \frac{6}{\pi} \text{ hours}$
 $t \approx 1.9 \text{ hours}$

General steps for this question:

- Find the θ that the plane travelled around earth in the given arc length.
- Use the θ with a ratio that compares the equivalent θ to the revolution that occurs over 6 hours to find t .

The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180° ? Assume the radius of the bicycle wheel is 13.6 in.



THINKING: pedal rotates 180° means that larger gear rotates 180° .

"Length" that the large gear rotates causes chain to move that "length" too. This also makes the small gear move this "length"; the angle the small gear rotates causes the wheel to move/rotate at this angle. The resulting distance travelled by the wheel is what is needed to be found.

MATHEMATICALLY: θ of pedal = θ of large gear.

- ② \rightarrow using θ of large gear + radius, find arclength.
arclength of large gear = arclength of small gear
- ③ \rightarrow using arclength + radius of small gear, find θ .
 θ of this small gear = θ of wheel
- ④ \rightarrow using θ of small gear, find arclength of wheel.

$$\theta = 180^\circ = \pi$$

$$r = 4.72 \text{ in}$$

$$a = ?$$

$$a = \theta r$$

$$a = \pi(4.72 \text{ in})$$

$$a = 4.72\pi \text{ in}$$

$$\theta = ?$$

$$r = 1.38 \text{ in}$$

$$a = 4.72\pi \text{ in}$$

$$\theta = \frac{a}{r}$$

$$\theta = \frac{4.72\pi \text{ in}}{1.38 \text{ in}}$$

$$\theta = \frac{236\pi}{69}$$

$$\theta = \frac{236\pi}{69}$$

$$r = 13.6 \text{ in}$$

$$a = ?$$

$$a = \theta r$$

$$a = \frac{236\pi}{69}(13.6)$$

$$a = \frac{16048\pi}{345} \text{ in}$$

① Read the question and identify things being "shared". While doing this, create a series of steps to see what the question wants you to find.

DOING THIS HELPS
VERBALIZE THE INFO
GIVEN TO YOU

② Execute steps as listed above.
Find arclength of large gear.

③ Find θ of small gear.

④ Find arclength of wheel

⑤ Write \therefore statement

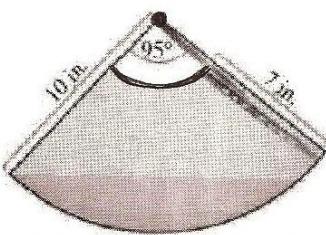
\therefore the bicycle will move approx 146.1 inches.

$$A = \frac{1}{2} r^2 \theta$$

NOTE: In this formula (or any formula that uses radius) make sure your value is quoted from being from the **CENTER** of your circle!!!!

①b) Sector Area

The total arm and blade of a single windshield wiper was 10 in. long and rotated back and forth through an angle of 95° . The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?



$$\begin{cases} \theta = 95^\circ \\ r = 10 \text{ in} \\ A = ? \\ 95^\circ = \frac{19\pi}{36} \end{cases}$$

$$A = \frac{1}{2} \left(\frac{19\pi}{36} \right) (10 \text{ in})^2$$

$$A = \frac{1}{2} \frac{19\pi}{36} (100 \text{ in}^2)$$

$$A = \frac{475\pi}{18} \text{ in}^2$$

① Find the area of the entire windshield.
REMEMBER, θ = RADIANS!

$$\theta = 95^\circ = \frac{19\pi}{36} \quad \leftarrow \text{YES, } \theta \text{ is the same!}$$

$$\begin{cases} r = 3 \text{ in} \\ A = ? \\ A = ? \end{cases}$$

$$A = \frac{1}{2} \left(\frac{19\pi}{36} \right) (3 \text{ in})^2$$

$$A = \frac{1}{2} \frac{19\pi}{36} (9 \text{ in}^2)$$

$$A = \frac{19\pi}{8} \text{ in}^2$$

② You MAY NOT use the 7 inches as a radius value. Refer to NOTE ↑

So, use $r=3$ and find the area of the windshield that doesn't get cleaned.

$$\frac{475\pi}{18} \text{ in}^2 - \frac{19\pi}{8} \text{ in}^2 = A \quad \leftarrow \text{that gets cleaned}$$

③ To find area that GETS CLEARED, subtract the two areas.

$$\frac{1729\pi}{72} \text{ in}^2 = A$$

④ Write ∴ statement

∴ The area of the windshield that gets cleaned is $\frac{1729\pi}{72} \text{ in}^2$ or approx. 75.44 in^2 .

②(a) Linear vs. Angular Speed KEY VARIABLES

a = distance travelled or arc length (inches, Kilometers, etc)

t = time (seconds, hours, days, years, etc.)

θ = amount of rotation(s) or angle (degrees, radians, rotations, revolutions)

r = radius or distance from CENTER of rotation (centimeters, inches, etc.)

v = linear speed = $\frac{\text{distance}}{\text{time}}$ (cm/sec, km/h, etc.)

w = angular speed = amount of rotation(s) (degrees/sec, radians/min, revolutions/min, etc.)

$$\frac{\text{"angle}}{\text{time}} \rightarrow \frac{\text{time}}{\text{revolutions/min, etc.}} \quad \text{rpm}$$

FORMULAS

$a = r\theta$	$v = \frac{a}{t}$	$w = \frac{\theta}{t}$	$v = rw$
---------------	-------------------	------------------------	----------

$v = rw$ is derived by combining the formula for linear speed with arclength.

$$v = \frac{a}{t} \text{ and } a = r\theta \Rightarrow v = \frac{r\theta}{t} = (r)\left(\frac{\theta}{t}\right) = rw \quad \therefore \text{linear speed is also } v = rw$$

CONVERSION FACTORS

$$1 = \frac{1000\text{m}}{1\text{km}} \quad \frac{12\text{ in}}{1\text{ ft}} \quad \frac{3.28084\text{ ft}}{1\text{ m}} \quad \frac{1\text{ mi}}{5280\text{ ft}} \quad \frac{1.609344}{1\text{ mi}}$$

$$1 = \frac{2\pi}{1\text{ rev}} \quad \frac{\pi}{180^\circ} \quad \frac{60\text{ min}}{1\text{ hr}} \quad \frac{60\text{ sec}}{1\text{ min}} \quad \frac{24\text{ h}}{1\text{ day}}$$

NOTE:

their reciprocals can be used. adjust factors according to what is needed.

(2b) Word Problem example (only conversion of units is happening)

- A car travels 3 miles and its tires make 2640 revolutions. What is the radius of the tire in inches?

$$\begin{aligned} a &= 3\pi r \\ \theta &= 2640 \text{ rev} \\ r &=? \leftarrow \text{inches} \end{aligned}$$

$$r = \frac{3\text{mi}}{2640\text{ rev}} \times \frac{5280\text{ ft}}{1\text{ mi}} \times \frac{12\text{ in}}{1\text{ ft}} \times \frac{1\text{ rev}}{2\pi}$$

$$r = \frac{(3)(5280)(12) \text{ in}}{(2640)(2\pi)} \quad \leftarrow \text{NOTE: only inches}$$

$$r = \frac{190080}{5280\pi} \text{ in} = \frac{36}{\pi} \text{ m}$$

∴ radius of tire
is $\frac{36}{\pi}$ inches

⑤ Write a statement

approx 11.5 inches.

① Identify what you know and what you need.

*State your desired units as well!!! *

② Write what you have using your formula

③ Begin a "conversion"

A simple line drawing of a train consisting of three rectangular cars connected by lines. The first car has two small circles at the bottom representing wheels. To the left of the train, the word "train" is written in a cursive script. Above the train, there is a wavy line representing clouds.

④ Simplify into one fraction. Cancel units and multiply all top #'s and all bottom #'s.

②) Word problems involving angular/linear speed

- A cylinder with a 2.5 ft radius is rotating at 120 rpm. rev/min

- (a) Give the angular velocity in rad/sec and in degrees per second.

- (b) Find the linear velocity of a point on its rim in mph.

$$= \frac{120 \text{ rev}}{60 \text{ sec}} \times \frac{2\pi}{1 \text{ rev}} = \frac{240\pi}{60 \text{ sec}} = 4\pi \text{ /sec}$$

∴ angular velocity is 4π /sec AND/OR $720^\circ/\text{sec}$

$$V = \frac{10\pi ft}{1sec} \times \frac{1mi}{5280ft} \times \frac{3600sec}{1hr}$$

∴ The linear velocity of a point on its rim in mph is approx 21.4 mph.

① As you read the ques., identify variables being given to you.

In part a), you are asked to find w. By definition, w is: an angle/amount of rotations time.

$$120 \text{ rpm} = \theta \text{ since } \frac{120 \text{ rev}}{\text{min}} = \frac{\theta}{\text{min}}$$

(2) Find w and convert your answer to units desired using "conversion train".

In part b), you are asked to find v . By definition, v can be $\frac{q}{t}$ or rw . In this ques.,

We know r and w . So, use
 $V = rw$ (Use w from part a)

③ Use formula to find v and a

then use "conversion train" to convert to desired units.

The radius of Earth is approximately 6400 km. It takes 23h 56m 4.1s for the Earth to rotate once on its axis. Find:

- angular speed of Earth in radians per day and radians per hour.
- linear speed at the North or South Pole.
- linear speed at Quito, Ecuador, a city on the equator.
- linear speed at Salem, Oregon (halfway from the equator to the North Pole)

a) $r = 6400 \text{ km}$

$$t = 23 \text{ h } 56 \text{ m } 4.1 \text{ s} = 23 \frac{56}{60} \text{ h } + \frac{4.1}{3600} \text{ h} \approx 23.93 \text{ h}$$

$$\omega = \frac{1 \text{ rev}}{23.93 \text{ h}} \times \frac{2\pi}{1 \text{ rev}} = \frac{2\pi}{23.93 \text{ h}} \text{ (radians/h)}$$

$$\omega = ? \text{ radians/h}$$

$$\frac{2\pi}{23.93 \text{ h}} = [0.26/\text{hr}]$$

$$\frac{2\pi}{23.93 \text{ h}} \times \frac{24 \text{ h}}{1 \text{ day}} = \frac{48\pi}{23.93 \text{ h}} \approx [6.3/\text{day}]$$

b) At NS pole, linear speed is 0 because it is on the axis so it is a fixed point that doesn't travel anywhere.
 $\omega = 0$, $r = 6400 \text{ km}$, $v = ?$, $a = 0$.

c) $r = 6400 \text{ km}$
 $\omega = ?$
 $v = ?$
 $v = r\omega$
 $v = 6400 \text{ km} / \left(\frac{2\pi}{23.93 \text{ h}} \right)$
 $v \approx 1680 \text{ km/hr}$

V = v
 \therefore Linear speed of the earth at Quito, Ecuador is approximately 1680 km/hr.

① LOOK for variable values as you are reading the question.

In part a, you are asked to find ω . By definition, ω is $\frac{\theta}{t}$.

θ can be an angle or revolutions or rotations.

In this example, we are told that the earth rotates ONCE every 23h... rotates once = 1 rev = θ

every 23h 56m 4.1s = t

② Convert your t so that it is expressed as ONE unit. Choose hours because that is needed to answer the question. Use equivalent ratios then ADD them.

③ After having converted your t into ONLY hours, sub-in t into your definition for ω and begin to solve. Use 1 rev as your θ !

④ Begin to convert to your desired unit(s). To get your answer in radians/h, simply multiply original by $\frac{1 \text{ rev}}{2\pi}$.

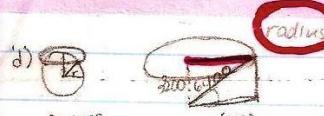
To get answer in radians/day, multiply previous answer by $\frac{24 \text{ h}}{1 \text{ day}}$

⑤ In part b, we are asked to find v at the north/south pole. By definition, $v = \frac{a}{t}$. At the north/south pole, the arclength of the earth rotating is zero. So $a = 0$. Therefore, linear speed at these 2 points would be zero!

⑥ In part c, we are asked to find v at the equator. By definition, $v = \frac{a}{t}$ or $v = r\omega$. In this question, we have already found a value for the angular speed, ω . The question also tells us that the Earth's radius is approx 6400km. Using ω and $r = 6400\text{km}$, we can find v . (use the ω quoted as being km/h because this is a common unit for linear sp)

part d is continued on the next page...

v can be defined as $\frac{a}{t}$ or $r\omega$. Since we know the ω , we can solve for r to find v .

d) 

$$r^2 = 6400^2 - 3200^2$$

$$r = \sqrt{6400^2 - 3200^2}$$

$$r = \sqrt{30720000} \text{ Km}$$

$$v = (\sqrt{30720000} \text{ Km}) \left(\frac{2\pi}{23,934,551 \text{ s}} \right)$$

↑
angular speed
of radians/hr

$$v = 1455.01 \text{ Km/hr}$$

∴ the linear speed
of the earth
at Salem, Oregon
is approx.
1455.01 Km/hr.

- ⑦ In part d, we are being asked to find v for a point located halfway between the north pole and equator.

Looking at the enlarged diagram drawn, we know that since $r=6400$, the point of the city is 2000km away from the equator.

To find the radius at this point, we need to think in a 3-dimensional way.

The diagonal of the Earth's radius to the point of the city can act as a hypotenuse. The vertical distance of the city can act as a leg. We can find the **radius** of the city (FROM "CENTER" of the Earth) using pythag. theory.

- ⑧ After having found the radius of this point, use that as r and use the ω found in part a) (km/h one) and solve for your linear speed.

→ Why find the **radius** instead of using one given? ←
we are worried about the linear speed of something that is higher up in the earth's sphere. Because of the curvature of the earth, the radius of the earth at that point is different from the earth's radius at the equator. HOWEVER, the angular speed remains the same because that city is still moving the same # of rotations. Linear speed is affected because of change in radius.

- When all units in a train cancel, and you are solving for w, the top is RADIANs.
- A tire with a 9-inch radius is rotating at 30mph. Find the angular velocity of a point on its rim. Express the result in radians per minute.

$$r = 9 \text{ in}$$

$$v = 30 \text{ mi}$$

$$\text{hr}$$

$$w = ? \text{ rad/min}$$

$$w = \frac{v}{r} = \frac{30 \text{ mi}}{\text{hr}} \times \frac{1}{9 \text{ in}}$$

$$w = \frac{30 \text{ mi}}{\text{hr}} \times \frac{1}{9 \text{ in}} \times \frac{12 \text{ in}}{1 \text{ ft}} \times \frac{5280 \text{ ft}}{1 \text{ mi}} \times \frac{1 \text{ hr}}{60 \text{ min}}$$

$$w = \frac{(30)(12)(5280)}{(9)(60)} \text{ min}$$

$$w = 3520 \text{ (radians)/min}$$

① State what is given to identify what you must find.

② Use equation to solve

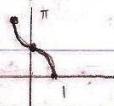
③ Begin conversion train

④ Quote answer (check units!!)

(4) How to sketch a transformed inverse trig function

• $y = \arccos(0.5x) - \pi$

parent:



Key points: $(1, 0)$
 $(0, \frac{\pi}{2})$
 $(-1, \pi)$

$$K = \frac{1}{2}, C = -\pi$$

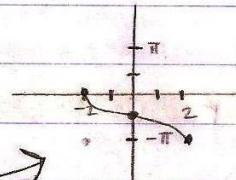
① Identify the parent inverse trig function to use its 3 key points to translate.

new points: $(2, -\pi)$

$(0, -\frac{\pi}{2})$
 $(-2, 0)$

② Identify transformations and transform points. Use mapping rule or graph of parent.

③ Graph your new points.



COMPUND ANGLE IDENTITIES

FINAL GRAPH

(5a) IDENTITIES

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

DOUBLE or HALF angle identities

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\cos(2A) = 1 - 2\sin^2 A$$

→ same as

$$\sin^2 A = \frac{1 - \cos(2A)}{2}$$

$$\cos(2A) = 2\cos^2 A - 1$$

→ same as

$$\cos^2 A = \frac{1 + \cos(2A)}{2}$$

$$\sin(2A) = 2\sin A \cos A$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

half angle
ID

$$\sin \frac{1}{2} A = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \frac{1}{2} A = \pm \sqrt{\frac{1 + \cos A}{2}}$$

OTHER IDENTITIES

ODD/EVEN SYMMETRY ID

If you need to use an ID but the trig function is squared, square the expanded version!

ODD

$$\begin{cases} \sin\theta = -\sin(-\theta) \\ \csc\theta = -\csc(-\theta) \\ \tan\theta = -\tan(-\theta) \\ \cot\theta = -\cot(-\theta) \end{cases}$$

↳ or $-\sin\theta = \sin(-\theta)$, etc.

$$\begin{aligned} \bullet \sin(-\frac{\pi}{6}) &\leftarrow \text{ODD} \\ &= -\sin(\frac{\pi}{6}) \end{aligned}$$

EVEN

$$\begin{cases} \cos\theta = \cos(-\theta) \\ \sec\theta = \sec(-\theta) \end{cases}$$

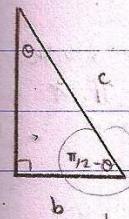
$$\bullet \sec(\frac{\pi}{2}) \leftarrow \text{EVEN}$$

$$= \sec(-\frac{\pi}{2})$$

$$\begin{aligned} \bullet \cos(-A-B) &\leftarrow \text{FACTOR } \Theta \\ \cos(-(A+B)) &\leftarrow \text{EVEN} \\ &= \cos(A+B) \end{aligned}$$

COFUNCTION ID

This identity has to do with the complementary angles within a right angle triangle



This angle is $90^\circ - \theta$ because the 2 must be complementary.

$$\begin{aligned} \sin\theta &= \frac{b}{c} = \cos(\frac{\pi}{2} - \theta) \\ \tan\theta &= \frac{b}{a} = \cot(\frac{\pi}{2} - \theta) \end{aligned}$$

sine and cosine are cofunctions,
cosecant and secant are cofunctions,
tangent and cotangent are cofunctions.

$$\begin{aligned} \bullet \sin(\frac{\pi}{6}) &= \cos(\frac{\pi}{2} - \frac{\pi}{6}) \leftarrow \text{DO LCD} \\ &= \cos(\frac{\pi}{3}) \\ \therefore \sin(\frac{\pi}{6}) &= \cos(\frac{\pi}{3}) \end{aligned}$$

$$\begin{aligned} \bullet \cot(2x + \frac{\pi}{4}) &= \tan(\frac{\pi}{2} - (2x + \frac{\pi}{4})) \\ &= \tan(\frac{\pi}{2} - 2x - \frac{\pi}{4}) \leftarrow \text{DO LCD} \end{aligned}$$

$$\begin{aligned} \bullet \sec(\frac{\pi}{8}) &= \csc(\frac{\pi}{2} - \frac{\pi}{8}) \leftarrow \text{DO LCD} \\ &= \csc(\frac{3\pi}{8}) \\ \therefore \sec(\frac{\pi}{8}) &= \csc(\frac{3\pi}{8}) \end{aligned}$$

$$\therefore \cot(2x + \frac{\pi}{4}) = \tan(\frac{\pi}{4} - 2x)$$

NOTE: When shifting by a similar graph, think of transformations!
So, if you want to move a graph to the LEFT,
+ **add** +. If you want to move it RIGHT **subtract**

HORIZONTAL TRANSLATIONS OF MULTIPLE PERIODS

Since periodic functions repeat in cycles, we can identify multiples of periods in the horizontal translation of a function to simplify them.

REMEMBER:

$$p = 2\pi \rightarrow \sin, \cos, \csc, \sec$$

$$p = \pi \rightarrow \tan, \cot$$

$\cos\left(\frac{7\pi}{3}\right)$ period = 2π so, let's shift this 2π to the right (that way we can subtract 2π .)

↓
shift by -2π

$$\cos\left(\frac{7\pi}{3} - 2\pi\right) = \cos\left(\frac{\pi}{3}\right)$$

LCD

$\tan(x - 4\pi)$ period = π so, let's shift this 4π to the left (that way we can ADD 4π .)

↓
shift by 4π

$$\tan(x - 4\pi + 4\pi) = \tan(x)$$

*we can use 4π because it's a multiple of π . *

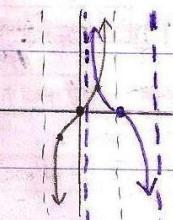
NOTE!
Shift
trig func.
You are
recording.
NOT
original

SHIFT OF A SIMILAR TYPE OF GRAPH

SIMILAR GRAPH PAIRS:

Sine \leftrightarrow cosine
Cosecant \leftrightarrow secant
Tangent \leftrightarrow cotangent

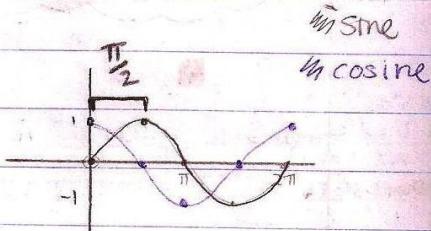
tangent
cotangent



THINK: How could we make the cotangent graph equal the tangent graph???

A: reflect in x-axis, shift left (or right) $\frac{\pi}{2}$

$$\tan\theta = -\cot(\theta \pm \frac{\pi}{2})$$



THINK: what horizontal shift to cosine would make it equal to sine?

A: shift right of $\frac{\pi}{2}$

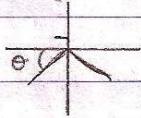
$$\text{Sine} = \cos(\theta - \frac{\pi}{2})$$

(1 version)
 $\sec\theta = \csc(\theta + \frac{\pi}{2})$

(MANY VARIATIONS!)

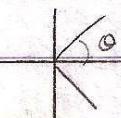
SAME RELATED ACUTE ANGLE → THINK CAST

• Sine negative



$$\sin(\pi + \theta) = \sin(2\pi - \theta)$$

• cosine positive



$$\cos\theta = \cos(2\pi - \theta)$$

• tangent neg.



$$\tan(\pi - \theta) = \tan(2\pi - \theta)$$

EXAMPLE of writing down several equivalent expressions using an angle.

• $\sec \frac{3\pi}{4}$

even symmetry:

$$\sec \frac{3\pi}{4} = \sec(-\frac{3\pi}{4})$$

cofunction:

$$\sec \frac{3\pi}{4} = \csc(\frac{\pi}{2} - \frac{3\pi}{4})$$

period shift:

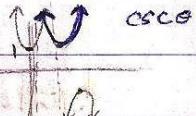
$$\sec \frac{3\pi}{4} = \sec(\frac{3\pi}{4} + \frac{2\pi}{4})$$

$$= \sec(\frac{3\pi + 8\pi}{4})$$

$$= \sec(\frac{11\pi}{4})$$

graph shift:

$$\sec \frac{3\pi}{4} = \csc(\frac{3\pi}{4} + \frac{\pi}{2})$$



related acute:

$$\sec \frac{3\pi}{4} = \sec(\frac{5\pi}{4})$$



where is secant negative?

REMEMBER: SHIFT THIS!

$$= \csc(\frac{3\pi}{4} + \frac{2\pi}{4})$$

$$= \csc(\frac{5\pi}{4})$$

NOTE: YOU MUST STATE THE "IDENTITY" YOU ARE USING!!!

Qb) How to find the exact value of a trig function using compound ID

• $\csc(\frac{11\pi}{12}) \rightarrow \frac{11\pi}{12} = \frac{2\pi}{12} + \frac{9\pi}{12}$

$$= \csc(\frac{2\pi}{12} + \frac{9\pi}{12})$$

$$= \csc(\frac{\pi}{6} + \frac{3\pi}{4})$$

$$= \frac{1}{\sin(\frac{\pi}{6} + \frac{3\pi}{4})}$$

OH NO! WE DON'T
HAVE A FORMULA
FOR CSC!!

① Change the input angle
into a sum/difference
of special angles.

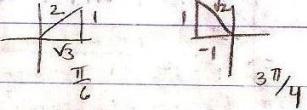
② Write them in your
trig function and
simplify.

③ Take the reciprocal
of the function

$$\csc\theta = \frac{1}{\sin\theta}$$

COMPOUND ID

$$\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{3\pi}{4}\right)$$



④ Now that we have sine, identifying an ID that can be used.

⑤ Use pictures to solve for the exact ratios of each.

(rationalize
at the
same time
to skip a
step later)

$$\rightarrow \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{4}{\sqrt{6} - \sqrt{2}} (\sqrt{6} + \sqrt{2}) = \frac{4\sqrt{6} + 4\sqrt{2}}{6 - 2} = \frac{4\sqrt{6} + 4\sqrt{2}}{4}$$

FINAL ANSWER!

$$= \sqrt{6} + \sqrt{2}$$

GENERAL STEPS!

① Find special Δ angles for input angle ③ Find ratios

② Identify ID

④ Simplify

EXTRA: If you see the following denominators in angles:

$\rightarrow \frac{\pi}{2} \rightarrow$ use a quadrantal picture

$\rightarrow \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3} \rightarrow$ use special Δ s

$\rightarrow \frac{\pi}{12} \rightarrow$ use compound ID

$\rightarrow \frac{\pi}{8} \rightarrow$ use half angle ID

EXTRA

How to find the exact value of a trig function using half angle ID

• $\tan -\frac{\pi}{8}$ denom 8 = half angle ID

(1) We don't have a half angle ID for tangent! Try to rewrite using sine/cosine.

QUOTIENT ID

$$\begin{aligned} \sin -\frac{\pi}{8} & \\ \cos -\frac{\pi}{8} & \end{aligned} \quad \left. \begin{array}{l} \text{HALF ID} \\ \hline \end{array} \right\} \quad \begin{aligned} \sin A &= \pm \sqrt{1 - \cos 2A} \\ \cos A &= \pm \sqrt{1 + \cos 2A} \end{aligned}$$

(2) Use the half angle ID for sine/cosine within the numerator/denom.

$$\frac{1}{\sqrt{1-\cos \frac{\pi}{4}}} = \pm \sqrt{\frac{1}{2}(1-\cos \frac{\pi}{4})} \quad * \text{the } \frac{1}{2} \text{ cancels on}$$

$$\pm \sqrt{\frac{1}{2}(1+\cos \frac{\pi}{4})} \quad \text{the top/bottom.}$$

* overall sign is Θ

(3) Simplify

Now also think about what the overall sign of your ratio must be.

$$\begin{aligned} &= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{\frac{1+\frac{\sqrt{2}}{2}}{2}}} = -\sqrt{\frac{2-\sqrt{2}}{2}} \quad \left. \begin{array}{l} \text{part whole} \\ \text{thing under} \\ \text{the root} \end{array} \right\} \\ &- \left(\sqrt{\frac{2-\sqrt{2}}{2}} \right) \quad * \text{the } \frac{1}{2} \text{ cancels again} \end{aligned}$$

* tan is Θ in quad IV

$$\begin{aligned} &= -\sqrt{\frac{(2-\sqrt{2})(2+\sqrt{2})}{2 \cdot 2}} = \sqrt{\frac{(2-\sqrt{2})^2}{4-2}} = \sqrt{\frac{(2-\sqrt{2})^2}{2}} = \frac{\sqrt{(2-\sqrt{2})^2}}{\sqrt{2}} \quad \begin{array}{l} \text{root} \\ \text{and} \\ \text{square} \\ \text{cancel} \end{array} \\ &= -\frac{(2-\sqrt{2}) \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = -\left(\frac{2\sqrt{2}-2}{2}\right) = -(\sqrt{2}-1) = -\sqrt{2}+1 \end{aligned}$$

$$\therefore \tan -\frac{\pi}{8} = -\sqrt{2}+1$$

(5c) How to find the exact values of half angles if given a ratio with a full angle

OLD! quad II

• $\cos A = -\frac{5}{12}$, $\frac{\pi}{2} < A < \pi$ FIND: $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$, $\tan \frac{1}{2}A$

$$\sin \frac{1}{2}A = \pm \sqrt{\frac{1-\cos A}{2}} \quad \text{HALF ID}$$

(1) To find $\sin \frac{1}{2}A$,

identify the ID you will be using.

$$= \pm \sqrt{\frac{1-\left(-\frac{5}{12}\right)}{2}} = \pm \sqrt{\frac{17}{24}} = \pm \frac{\sqrt{17}}{\sqrt{24}} = \pm \frac{\sqrt{408}}{24}$$

$$\pm \frac{2\sqrt{102}}{24} = \pm \frac{\sqrt{102}}{12}$$

(3) To decide sign, create a new inequality.

(2) Sub-in your ratio value for $\cos A$ and simplify.

$$\therefore \left\{ \frac{\sqrt{102}}{12} \right\} = \sin \frac{1}{2}A \quad \left\{ \frac{\pi}{4} < \frac{1}{2}A < \frac{\pi}{2} \right\} \text{NEW! quad I}$$

④ Shortcut to completing $\cos \frac{1}{2}A$ and $\tan \frac{1}{2}A$ with ratio of $\sin \frac{1}{2}A$

SHORTCUT: After finding the ratio of one trig function, you can use it to find the others by using the NEW domain inequality and the NEW ratio. ($\sin \frac{1}{2}A$)

WE KNOW: $\frac{\pi}{4} < \frac{1}{2}A < \frac{\pi}{2}$ (quadrant I)

$$\sin \frac{1}{2}A = \frac{\sqrt{102}}{12} = \frac{y}{r}$$

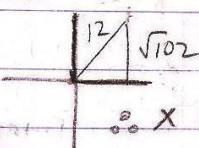
⑤ Find x

if $y = \sqrt{102}$ then $\sqrt{102}^2 - (\sqrt{102})^2 = x$

$$\pm \sqrt{42} = x$$

⑥ Determine if x is \pm

To determine if x is \pm , draw a picture using what you already know.



$\therefore x$ is \oplus

⑦ Use

all your
new values
to finish
finding your ratios.

NEW

$$\cos \frac{1}{2}A = \frac{x}{r} = \frac{\sqrt{42}}{12}$$

↓
ANSWER

$$\tan \frac{1}{2}A = \frac{y}{x} = \frac{\sqrt{102}}{\sqrt{42}}$$

$$= \frac{\sqrt{17} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$

$$= \frac{\sqrt{119}}{7}$$

← ANSWER

think of $\frac{102}{42} = \frac{17}{2}$

⑥ Proofs:

a) List of strategies to try

- get everything to be sine or cosine because out of our new identities, we don't have any that use cot, sec, csc.
- when you see values within your angle (the input area), look for double angles or expressions that can be simplified using other IDs like symmetry, graph shift, cofunction.
- look to arrange terms in ways that you can make them identical to compound angle ID, double ID, etc.

b) Examples of proving identities

$$\bullet \sin 4x = 2\cot(2x)\sin^2(2x)$$

$$RS = 2\cot(2x)\sin^2(2x)$$

$$= \frac{2\cos(2x)\sin^2(2x)}{\sin(2x)}$$

→ QUOTIENT ID

↓ cancel $\sin(2x)$

$$= 2\cos(2x)\sin(2x)$$

$$= 2\cos A \sin A$$

① Choose a side to begin with.

② Try to make everything sin/cos

③ Try to get a single expression because you may be able to use a double angle ID.

$$-\sin 2A$$

$$= \sin(2 \cdot 2x)$$

$$= \sin 4x \quad \therefore LS = RS$$

(4) Use the $\sin(2A)$ double angle ID

$$\bullet \tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = 2\sec(2x)$$

$$LS = \tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right)$$

$$= \tan\frac{\pi}{4} + \tan x + \tan\frac{\pi}{4} - \tan x$$

$$1 + \tan\frac{\pi}{4}\tan x \quad 1 + \tan\frac{\pi}{4} - \tan x$$

$$= \frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x}$$

$$= \frac{(1 + \tan x)(1 + \tan x) + (1 - \tan x)(1 - \tan x)}{(1 + \tan x)(1 + \tan x)}$$

$$= \frac{1 + 2\tan x + \tan^2 x + 1 - 2\tan x + \tan^2 x}{1 + \tan^2 x}$$

$$= \frac{2 + 2\tan^2 x}{1 - \tan^2 x} = \frac{2(1 + \tan^2 x)}{1 - \tan^2 x}$$

$$\frac{2\sec^2 x}{1 - \sin^2 x} \quad - \frac{2\sec^2 x}{\cos^2 x}$$

$$\frac{2\sec^2 x \cos^2 x}{\cos^2 x - \sin^2 x} = \frac{2\sec(2x)}{\cos(2x)} \quad \therefore LS = RS$$

(1) Choose a side to work

with

(2) Use the compound angle ID

(3) Simplify, $\tan\frac{\pi}{4} = 1$

(4) LCD

(5) Simplify

(6) Use $\tan^2 \theta + 1 = \sec^2 \theta$
and quotient ID

(7) Multiply by reciprocal
AND use double angle ID on
 $\cos^2 x - \sin^2 x$

(8) Cancel $\sec^2 x$
and $\cos^2 x$
then use reciprocal ID

STRATEGY

multiply by conjugate when stuck!!

ALSO, DON'T FORGET ABOUT THE "OTHER"
IDENTITIES! (similar shape, shift by period, etc.)

When solving, DON'T DIVIDE to cancel! FACTOR DD

Solving Trig Equations

(a) Ex. of one that requires factoring. + general solution

$$3\sec x \sin x - 2\sqrt{3} \sin x = 0$$

$$\sin x (3\sec x - 2\sqrt{3}) = 0$$

$$\sin x = 0 \quad y$$

$$+$$

$$x_1 = 0$$

$$x_2 = \pi$$

$$x_3 = 2\pi$$

$$x_n = 0 + \pi n, n \in \mathbb{Z}$$

$$3\sec x - 2\sqrt{3} = 0$$

$$\sec x = \frac{2\sqrt{3}}{3}$$

UNRATIONALIZE

$$\sec x = \frac{2}{\sqrt{3}} = \frac{r}{x}$$

① Common factor

② Make each factor = 0
Then solve for x.

③ To state general solutions
for questions where you
factored, you must
have different sets of
sequences.

$$x_1 = \frac{\pi}{6}$$

$$x_4 = \frac{11\pi}{6}$$

$$x_5 = \frac{17\pi}{6}$$

$$x_k = \frac{\pi}{6} + 2\pi k$$

and $k \in \mathbb{Z}$

$$x_k = \frac{11\pi}{6} + 2\pi k$$

use different
subscript
because it's
a different
set of sequences.

(b) Input angle with a
k-value

$$\cot(3x) = -4 = 2\cot(3x) + 7$$

$$-11 = \cot(3x)$$

$$-11 = \cot \theta$$

$$-\frac{1}{11} = \tan \theta$$

$$\text{STO } \rightarrow \theta_r = \tan^{-1}(-\frac{1}{11}) \approx 0.09\dots$$

$$\text{STO } \rightarrow \theta_1 = \pi - \theta_r \approx 3.05$$

$$\theta_2 = 2\pi - \theta_r \approx 6.19$$

① Bring all Hs to one side and isolate the trig function

② We don't know how to solve these if there is a k-value.

We can't rely on ratios + inverses.

SO, $\theta_1 + \theta = 3x$

③ Find θ_1 then θ_2 and θ_3

④ Use θ_1 and θ_2 to do a replacement

⑤ To state general solutions for trig equations that have a k-value, to get more solutions, + period!

$$\frac{\theta_1}{3} = x$$

$$\frac{\theta_2}{3} = x$$

$$1.02 = x$$

$$2.06 = x$$

$$x_n = 1.02 + \frac{\pi}{3}n, n \in \mathbb{Z}$$

Here, we are adding the period. BECAUSE period of cot = π
 $\frac{\text{period}}{\text{k-value}} = \frac{\pi}{3}$

$$P = \frac{\pi}{\frac{1}{3}} = \frac{\pi}{3}$$

EXTRA! Solving a trig equation

$$6\cos(x+30^\circ)\sin(x+30^\circ) + 5 = 2 \quad (x \in [-270^\circ, 180^\circ])$$

$$3 \cdot 2\cos(x+30^\circ)\sin(x+30^\circ) = -3$$

double ID

$$3 \cdot \sin(2(x+30^\circ)) = -3$$

$$\sin(2(x+30^\circ)) = -1$$

$$\theta =$$

$$\sin \theta = \frac{-1}{1} = \frac{y}{r}$$

$$\theta = 270^\circ$$

$$\theta_n = 270^\circ + 360^\circ n, n \in \mathbb{Z}$$

$$\frac{2(x+30^\circ)}{2} = \frac{270^\circ + 360^\circ n}{2}$$

$$x+30^\circ = 135^\circ + 180^\circ n$$

$$\rightarrow x_n = 105^\circ + 180^\circ n$$

$$x_1 = 105^\circ$$

$$x_2 = -75^\circ \quad (n = -1)$$

$$x_3 = -255^\circ \quad (n = -2)$$

① Rewrite to use an ID.

② Do a replacement

③ Find values for θ

④ Create a general sequence of solutions because a replacement was done

⑤ Do the replacement of the general solution.

This is to create a general sequence for solutions for x .

⑥ Use your x_n sequence to find all solutions that exist within the domain.