

CONGRATULATIONS! YOU SURVIVED! WELCOME TO CALCULUS!



## Unit 1 - Rates & Limits

### (1a) What is a limit?

A limit is a function that describes what a function approaches as it gets closer to a point.

This is useful when you can't work something out directly but you can evaluate what is happening around the point of discontinuity, (FOR EX. A HOLE OR VA)

### What is the notation?

"The limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$ !"

$$\lim_{x \rightarrow a} f(x) = L$$

### What are the conditions for a limit to exist?

(from LS)

- $\lim_{x \rightarrow a^-} f(x)$  MUST EXIST

"EXIST" = #

EXIST  $\neq \pm\infty$

(from RS)

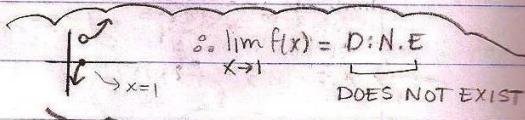
- $\lim_{x \rightarrow a^+} f(x)$  MUST EXIST

- $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$  If this is true,  $\lim_{x \rightarrow a} f(x)$  exists.

### The limit WILL NOT EXIST if:

- there is a jump in the graph

WHY: condition #3 from above is being broken.



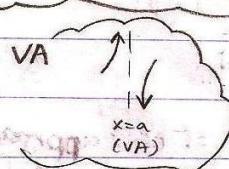
- there is a one-sided graph

WHY: condition #1 / #2 from above is being broken.

there is no LS for the graph to approach  $a$ .

- approaching  $\pm\infty$  because of a VA

WHY: conditions #1 / #2 / #3 from above are being broken.



#1 If, when  $x \rightarrow a^-$ ,  $y \rightarrow \infty$ , the limit does not exist because  $\infty$  does not exist!

#2 If, when  $x \rightarrow a^+$ ,  $y \rightarrow -\infty$ , the limit does not exist because  $-\infty$  does not exist!

#3 This can't be used unless #1 and #2 are satisfied.

### NOTE:

In a limit, if there is a  $\pm\infty$  as the value  $x$  IS APPROACHING, the output of the limit is an HA.

In a limit, if the OUTPUT is a  $\pm\infty$ , the INPUT of the limit is a VA.

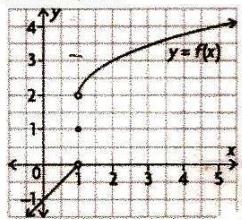
\* It is possible for an HA to be present on one side of the graph!

\* CONCEPTS FROM ①a) WILL BE USED IN EXAMPLES BELOW \*

**①b) finding one sided limits (and limits) from a piecewise graph**

- 

Find the following limits.



a.  $\lim_{x \rightarrow 5^-} f(x)$

b.  $\lim_{x \rightarrow 5^+} f(x)$

c.  $\lim_{x \rightarrow 1^-} f(x)$

d.  $\lim_{x \rightarrow 1^+} f(x)$

e.  $\lim_{x \rightarrow 5} f(x)$

f.  $\lim_{x \rightarrow 1} f(x)$

- a) We are asked to find the output value for limit functions by looking at the graph. a) is a one sided limit because it is specifically asking for what  $f(x)$  approaches as  $x$  approaches 5 FROM THE LEFT.

- $\lim_{x \rightarrow 5^-} f(x) = 4$

- b) we are asked to do the same as we did in part a) EXCEPT, we must look at what  $f(x)$  approaches as  $x$  approaches 5 FROM THE RIGHT.

- $\lim_{x \rightarrow 5^+} f(x) = 4$

- c) What does  $f(x)$  approach as  $x$  approaches 1 FROM THE LEFT?

- $\lim_{x \rightarrow 1^-} f(x) = 0$

- d) what does  $f(x)$  approach as  $x$  approaches 1 FROM THE RIGHT?

- $\lim_{x \rightarrow 1^+} f(x) = 2$

- e) To find the limit of  $x$  approaching 5 from BOTH LEFT AND RIGHT side, we must use condition #3 found under what conditions are there for a limit to exist. Since BOTH our one sided limits is 4, the limit exists.

- $\lim_{x \rightarrow 5} f(x) = 4$

- f) To find the limit of  $x$  approaching 1 from both sides, we must use condition #3 again. Since the one sided limits are different, the limit does not exist. (Another indicator is at  $x=1$ , there are jumps in the graph). So, we say:

- $\lim_{x \rightarrow 1} f(x) = \boxed{\text{D.N.E}}$

MUST BE IN  
 $\frac{0}{0}$  FORM

ALL LIMITS CAN BE EVALUATED  
 USING THESE TECHNIQUES  
 WHEN THIS RULE IS TRUE

(2b) Examples of finding limits that are in indeterminate form

METHOD: FACTOR/EXPAND

$$\bullet \lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + 5x - 6}{(x-2)} \rightarrow f(x)$$

ROUGH:  $f(x) = x^3 - 3x^2 + 5x - 6$

RRT:  $\pm 6, \pm 3, \pm 2, \pm 1$

FACTOR THEOREM:  $f(2) = 0 \therefore (x-2)$  is a factor

Synthetic division:

$$\begin{array}{r} 2 \\ \boxed{1} -3 5 -6 \\ \hline 2 -2 6 \\ \hline 1 -1 3 0 \end{array}$$

$$= x^2 - x + 3 \leftarrow \text{NOT FACTORABLE}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x^2 - x + 3)}{(x-2)}$$

CANCELLED  
PROBLEM!!

① Try direct substitution  
into function  $f(x)$

$$\text{rough: } f(2) = 8 - 12 + 10 - 6$$

$$= \frac{0}{0} \rightarrow \text{indeterminate form!}$$

② Factor using  
Rational Root theorem  
and factor theorem.  
Then, use synthetic division

$$= \lim_{x \rightarrow 2} (x^2 - x + 3) \quad \text{D.S.} \quad = (2)^2 - 2 + 3 = 4 - 2 + 3$$

③ Sub-in to limit (factors)

④ D.S. → do direct substitution  
NOW!

⑤ Write ∴ statement

$$\therefore \lim_{x \rightarrow 2} \frac{(x-2)(x^2 - x + 3)}{(x-2)} = 5$$

METHOD: LCD

$$\bullet \lim_{x \rightarrow 5} \frac{\left(\frac{1}{x}\right) - \left(\frac{1}{5}\right)}{(x-5)} = \lim_{x \rightarrow 5} \frac{\frac{5-x}{5x}}{(x-5)}$$

① Try direct substitution  
→ we get  $\frac{0}{0}$

$$= \lim_{x \rightarrow 5} \frac{-\frac{1}{5x}}{(x-5)} = \lim_{x \rightarrow 5} \frac{-\frac{1}{5x} \cdot \frac{1}{(x-5)}}{(x-5)}$$

② Do an LCD in the numerator

$$= \lim_{x \rightarrow 5} \frac{-\frac{1}{5x}}{\frac{5x - 25}{5x(x-5)}} \quad \text{D.S.} \quad = \frac{-1}{5(5)} = \frac{-1}{25}$$

③ Factor out a negative from  $(5-x)$  then multiply by the reciprocal

$$\therefore \lim_{x \rightarrow 5} \frac{\left(\frac{1}{x}\right) - \left(\frac{1}{5}\right)}{(x-5)} = -\frac{1}{25}$$

④ Write ∴ statement

## METHOD: RATIONALIZE/UNRATIONALIZE

$$\bullet \lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}$$

① Try direct substitution  
→ we get  $\frac{0}{0}$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{x+4})^2 - (2)^2}{x(\sqrt{x+4} + 2)} = \lim_{x \rightarrow 0} \frac{(x+4) - 4}{x(\sqrt{x+4} + 2)}$$

② Notice the radical.  
Multiplying by conjugate  
to unrationalize  
DON'T EXPAND  
DENOMINATOR!

$$= \lim_{x \rightarrow 0} \frac{x}{x\sqrt{x+4} + 2} \quad \text{SEE: } x\text{'s CANCELLED!}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4} + 2} \quad \text{D.S.} \quad = \frac{1}{\sqrt{0+4} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \left(\frac{1}{4}\right)$$

$$\therefore \lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} = \frac{1}{4}$$

③ Write ∴ statement

## METHOD: CHANGE OF VARIABLE

$$\bullet \lim_{x \rightarrow 1} \frac{\sqrt[3]{x-1}}{\sqrt[4]{x-1}} = \lim_{x \rightarrow 1} \frac{x^{1/3}-1}{x^{1/4}-1}$$

① Try direct substitution  
→ we get  $\frac{0}{0}$

$$\text{THINK: } \begin{cases} \frac{1}{3} \times 12 = 12/3 = 4 \\ \frac{1}{4} \times 12 = 12/4 = 3 \end{cases} \quad \left. \begin{array}{l} 12 \text{ is special} \\ \text{here} \end{array} \right\}$$

② Notice the ~~WEIRD~~/CRAY CRAY roots in the limit. Think of a variable + exponent that would help get rid of fractional ones.

$$\boxed{\text{let } u^{12} = x}$$

③ Do a let statement and sub-in  $u$ .

$$\lim_{u \rightarrow 1} \frac{(u^{1/2})^{1/3}-1}{(u^{1/2})^{1/4}-1} = \lim_{u \rightarrow 1} \frac{(u^{4/12}-1)}{(u^{3/12}-1)}$$

$$\left. \begin{array}{l} x \rightarrow 1 \\ u^{12} \rightarrow 1 \\ u \rightarrow 1 \end{array} \right\} \quad \begin{array}{l} \text{Also sub-in} \\ \text{to find this} \end{array}$$

$$= \lim_{u \rightarrow 1} \frac{(u^{2/12})(u^{1/12})(u^{-1/12})}{(u^{1/12})(u^{2/12}+u^{1/12})}$$

④ FACTOR

$$= \lim_{u \rightarrow 1} \frac{(u^{2/12})(u^{1/12})}{(u^{1/12})(u^{2/12}+u^{1/12})} \quad \text{D.S.} \quad = \frac{(1^{2/12})(1^{1/12})}{(1^{2/12}+1^{1/12})}$$

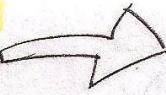
$$= \boxed{\frac{4}{3}}$$

⑤ Write ∴ statement

$$\therefore \lim_{x \rightarrow 1} \frac{\sqrt[3]{x-1}}{\sqrt[4]{x-1}} = \frac{4}{3}$$

NOTE: YOU DON'T HAVE TO GO BACK TO  $x$ 's.

How to decide  
which side is!  
left/right!



approaching  
FROM LS = negative version,  $x < \square$   
approaching  
FROM RS = positive version,  $x > \square$

### LIMIT OF AN

#### EXAMPLE: ABSOLUTE VALUE FUNCTION

(0/0)

$$\lim_{x \rightarrow 0^-} \frac{5x - |2x|}{4x + |x|} = \lim_{x \rightarrow 0^-} \frac{5x - (-2x)}{4x + (-x)}$$

$$= \lim_{x \rightarrow 0^-} \frac{5x + 2x}{4x - x} = \lim_{x \rightarrow 0^-} \frac{7x}{3x} \quad \boxed{D} = \left(\frac{7}{3}\right)$$

$$\therefore \lim_{x \rightarrow 0^-} \frac{5x - |2x|}{4x + |x|} = \frac{7}{3}$$

EASY →

MORE COMMON →

(0/0)

$$\lim_{x \rightarrow -2} \frac{(x+2)^3}{|x+2|} = \begin{cases} \lim_{x \rightarrow -2^+} \frac{(x+2)^3}{(x+2)}, & x \geq 2 \\ \lim_{x \rightarrow -2^-} \frac{(x+2)^3}{-(x+2)}, & x < 2 \end{cases}$$

$$(+) \lim_{x \rightarrow -2^+} \frac{(x+2)^3}{(x+2)}$$

$$= \lim_{x \rightarrow -2^+} (x+2)^2 \quad \boxed{D.S.} \quad = (-2+2)^2 \\ = 0^2 \\ = \boxed{0}$$

$$(-) \lim_{x \rightarrow -2^-} \frac{(x+2)^3}{-(x+2)}$$

$$= \lim_{x \rightarrow -2^-} -(x+2)^2 \quad \boxed{D.S.} \quad = -(-2+2)^2 \\ = -(0)^2 \\ = \boxed{0}$$

$$\therefore \lim_{x \rightarrow -2} \frac{(x+2)^3}{|x+2|} = 0$$

① Notice that since we are being asked to find the limit of the function as  $x$  approaches 0 FROM THE LEFT, we must take the NEGATIVE version of the absolute values.

② Make the absolute value factors  $\ominus$  and then CLT and simplify.

③ Write  $\therefore$  statement

① Create a piecewise function for the positive/negative versions of the absolute value.

REMEMBER: you will now have 2 one-sided limits! Choose which approaches from LS and RS using note bubble on top of page.

② Evaluate each one-sided limit separately.

③ If both one-sided limits are the same, write  $\therefore$  for the limit. IF NOT, say D.N.E.

\* NOTE: FOR DOMAIN RESTRICTIONS, make one inequality "equal to".

NOTE: When you get D.N.E or  $\#$ , say that limit overall D.N.E but then evaluate each one  $\overset{0}{\text{side}}$  limit. If one-sided limits are the same, then record it as an overall TO BE SPECIFIC.

$$\lim_{x \rightarrow \infty} f(x) = L$$

### (3) How to evaluate (indeterminate) limits at infinity (end behaviour / HA)

#### INFINITE LIMITS THEOREM

$$\lim_{x \rightarrow \pm\infty} \left( \frac{\#}{x^n} \right) = 0 \text{ if } n > 1$$

The limit of function  $\square$   
over  $x$  to the power of  $\square$   
equals zero (approaches zero)

$$\lim_{x \rightarrow \infty} \frac{7x^3 + x + 12}{2x^3 - 5x}$$

① Using limit law #6, let's split up  
this limit to understand what is  
happening.

$$= \lim_{x \rightarrow \infty} \frac{7x^3 + x + 12}{2x^3 - 5x} = \frac{\infty}{\infty}$$

② Think of the graphs of the  
numerator + denominator  
to visualise the end behav.  
and where they go as  $x \rightarrow \infty$

rough:  $7x^3 + x + 12$        $2x^3 - 5x$   
 ① L.C      ④ L.C  
 cubic      cubic

③ Sub-in the values of these  
limits to see another  
**INDETERMINATE FORM!**

$\frac{\infty}{\infty}$  example

$$\lim_{x \rightarrow \infty} \frac{7x^3 + x + 12}{2x^3 - 5x} \left( \frac{1}{x^3} \right)$$

④ (Evaluating begins)  
Notice that the highest  
power (overall) is the power 3.

\* Divide the top/bottom by  
the highest power \*  
and split the division  
amongst terms.

⑤ Cancel and then look to  
use the infinite limits  
theorem.

⑥ After cancelling by saying  
which ones approach/equal  
zero, the remaining value(s)  
is your limit.

$$= \lim_{x \rightarrow \infty} \frac{\left( \frac{7x^3 + x + 12}{x^3} \right)}{\left( \frac{2x^3 - 5x}{x^3} \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{\left( 7 + \frac{1}{x^2} + \frac{12}{x^3} \right)}{\left( 2 - \frac{5}{x^2} \right)} = \frac{7}{2}$$

$$\therefore \lim_{x \rightarrow \infty} \frac{7x^3 + x + 12}{2x^3 - 5x} = \frac{7}{2}$$

THINK: It makes sense

that the value of these limits is a #  
because when  $x \rightarrow \infty$ , it is approaching an HA.  
The # you find is the HA value.

⑦ Write  $\infty$  statement

EXAMPLE  
WHERE  
END BEHAVIOUR

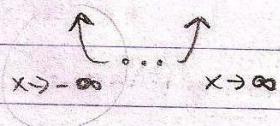
SKETCH  
NEEDS TO

B E T D O N E

$$\bullet \lim_{x \rightarrow -\infty} (x^4 + 7x^2 + 3)$$

x → -∞

highest power = 4 ∴ quartic  
parent is  $f(x) = x^4$



① Notice that there are NO x's on the bottom, so we can't divide by the highest power

② Notice we can't factor this (even as a pseudo)

③ Look at highest power to determine parent then do an END BEHAVIOUR sketch.

④ Even though limit D.N.E, be specific.

$$\therefore \lim_{x \rightarrow -\infty} (x^4 + 7x^2 + 3) = \text{D.N.E}$$

$$\therefore \lim_{x \rightarrow -\infty} (x^4 + 7x^2 + 3) = \infty \quad \downarrow \text{be specific}$$

(3b) How to evaluate (indeterminate) infinite limits (VA)  $\lim_{x \rightarrow a} f(x) = \pm\infty$

$$\bullet \lim_{x \rightarrow 3} \frac{x^2 + 5x + 1}{x^2 - 2x - 3}$$

① Notice that the top cannot be factored so we won't be able to cancel the problem.

$$\lim_{x \rightarrow 3^-} \frac{x^2 + 5x + 1}{x^2 - 2x - 3} = \frac{(2,999)^2 + 5(2,999) + 1}{(2,999)^2 - 2(2,999)} \quad \text{② The only other thing left to do here is to try to evaluate this numerically.}$$

$$\lim_{x \rightarrow 3^+} \frac{x^2 + 5x + 1}{x^2 - 2x - 3} = \frac{(3,001)^2 + 5(3,001) + 1}{(3,001)^2 - 2(3,001) - 3}$$

→ -∞

$$\therefore \lim_{x \rightarrow 3} \frac{x^2 + 5x + 1}{x^2 - 2x - 3} = \text{D.N.E}$$

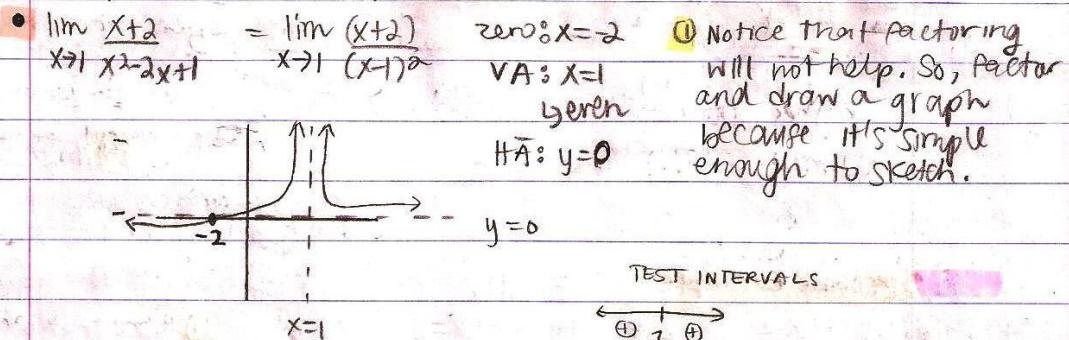
④ Write statement

③ Notice that the overall limit D.N.E and even when we try to be specific about which infinity the overall limit approaches, the different sides approach different infinities.

\* When finding each one-sided limit, if they were the same infinity, we would state it. This will be shown in the next example.

EXAMPLE OF D.N.E TYPE

Example of one that does (not) exist but is infinite OVERALL



Both one-sided limits don't exist  
so the overall limit doesn't exist.  
BUT, we can record a specific answer.

(2) Look at behaviour  
at  $x=1$   
BOTH SIDES APPROACH  
 $+\infty$ .

$$\therefore \lim_{x \rightarrow 1} \frac{x+2}{x^2-2x+1} = +\infty$$

(3) Write a statement

(3c)  $\infty - \infty$  example

$$\lim_{x \rightarrow \infty} \sqrt{x^2+4x+1} - x$$

(1) The only thing we can do here is multiply by the conjugate.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+4x+1} - x}{1} \cdot \frac{\sqrt{x^2+4x+1} + x}{\sqrt{x^2+4x+1} + x}$$

WHY IS THIS  $\infty - \infty$ ?

$$\begin{aligned} & \lim_{x \rightarrow \infty} \sqrt{x^2+4x+1} - x \quad \text{LIMIT LAWS} \\ &= \lim_{x \rightarrow \infty} \sqrt{x^2+4x+1} - \lim_{x \rightarrow \infty} x \quad \text{evaluate each} \\ &= \infty - \infty \end{aligned}$$

↑ INDETERMINATE

$$= \lim_{x \rightarrow \infty} \frac{x^2+4x+1-x^2}{\sqrt{x^2+4x+1} + x} = \lim_{x \rightarrow \infty} \frac{4x+1}{\sqrt{x^2+4x+1} + x} \quad (2) \text{ Are we stuck?}$$

NOTE: For questions involving roots, take the highest power and then create an equivalent monomial using a root.

NO! Use what you know about limits going to  $\infty$  (H.A. type)  
Multiply by the highest power.

This allows us to simplify under the root.

$$= \lim_{x \rightarrow \infty} \frac{4x}{\sqrt{x^2}} + \frac{1}{\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{4x}{x} + \frac{1}{x} \quad (3) \text{ Simplify + apply the infinite limits theorem.}$$

$$= \frac{\sqrt{x^2+4x+1}}{\sqrt{x^2}} + \frac{x}{\sqrt{x^2}}$$

$$= \sqrt{\frac{x^2+4x+1}{x^2}} + \frac{x}{x}$$

HIGHEST power:  $x$

$\exists \sqrt{x^2}$   
we can  
write as  
just  $x$   
because our  
limit is  
going to  $+\infty$ .  
(If it was going  
to  $-\infty$ , we would  
have a  $-x$ )

cont'd...

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x^2}}{\sqrt{1 + \frac{4}{x^2}}} = \lim_{x \rightarrow \infty} \frac{4}{\sqrt{1+1}} = \frac{4}{2} = 2$$

$$\therefore \lim_{x \rightarrow \infty} \sqrt{x^2 + 4x + 1} - x = 2$$

(4) Write  $\lim$  statement

## (3d) Piecewise Example

FIND:

$$\bullet f(x) = \begin{cases} \frac{x-2}{x-1}, & x \leq 8 \\ 56, & x=8 \\ \frac{1}{x-1}, & x > 8 \end{cases}$$

POINT  $x=8$   
 $y=56$   
just tells us  
that split occurs  
at  $x=8$   
we don't  
use it to  
find limit

- (1) Notice that because of the domain values given, the point at which the graph splits is  $x=8$ .

So we must use one-sided limits. If they match, the limit at 8 exists.

use piece #1

$$\lim_{x \rightarrow 8^+} \frac{x-2}{x-1} \quad \text{(P.S.)} = \frac{8-2}{8-1} = \frac{6}{7}$$

use piece #2

$$\lim_{x \rightarrow 8^-} \frac{1}{x-1} \quad \text{(D.S.)} = \frac{1}{8-1} = \frac{1}{7}$$

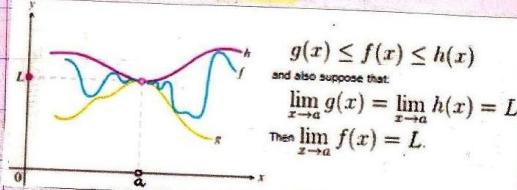
$$\lim_{x \rightarrow 8^+} f(x) \neq \lim_{x \rightarrow 8^-} f(x) \quad \therefore \lim_{x \rightarrow 8} f(x) = \text{D.N.E}$$

(2) Do the one-sided limits match?

(3) Write  $\lim$  statement

## (3e) Squeeze Theorem

NOTE: When using squeeze th., intervals, if lowest side of -1 is squared, the left interval WON'T be 1, but zero because if sine/cosine is squared, lowest output = 0.



66 If the graph  $h(x)$  and  $g(x)$  converge at a point,  $f(x)$  also converges at this point. Therefore, the limit at this point for  $h(x)$  and  $g(x)$  are the same and is also the same for  $f(x)$ .  $\Rightarrow$

This theorem is useful for sinusoidal functions like sine and cosine!! \*insert eyeroll here because sine/cosine appear everywhere\*

"SETUP" to use on sine/cosine:

$$\lim_{x \rightarrow a} -1 \leq \lim_{x \rightarrow a} \sin x \leq \lim_{x \rightarrow a} 1 \quad \text{OR} \quad \lim_{x \rightarrow a} -1 \leq \lim_{x \rightarrow a} \cos x \leq \lim_{x \rightarrow a} 1$$

$f(x)$   
↓

•  $\lim_{x \rightarrow -\infty} \frac{\cos x}{3x} \quad \lim_{x \rightarrow -\infty} -1 \leq \lim_{x \rightarrow -\infty} \cos x \leq \lim_{x \rightarrow -\infty} 1$

① Create an inequality  
Using squeeze theorem

NOTE:

• Flip the  
signs when  
we divide  
by  $x$

• Because our  
limit  $\rightarrow -\infty$

$$\lim_{x \rightarrow -\infty} \frac{-1}{3x} \geq \lim_{x \rightarrow -\infty} \cos x \geq \lim_{x \rightarrow -\infty} \frac{1}{3x}$$

② Manipulate the  
middle,  $f(x)$ , so  
that it resembles

The function in the  
limit. (What you  
do to the middle,  
you do everywhere)

$$0 \geq \lim_{x \rightarrow -\infty} \frac{\cos x}{3x} \geq 0$$

$$\therefore \lim_{x \rightarrow -\infty} \frac{\cos x}{3x} = 0$$

③ Evaluate LS + RS  
using infinite limits  
theorem.

\* BOTH ONE-SIDED LIMITS MUST BE EQUAL

④ When LS + RS = , the  
value is also the  
limit of  $f(x)$ .

∴ statement

### ③(f) Properties of Infinite Limits

$\lim_{x \rightarrow a} f(x) = \pm\infty$

Let  $c$  and  $L$  be real numbers and let  $f$  and  $g$  be functions such that

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L$$

1. Sum or difference:  $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty \quad \infty \pm L$

2. Product:  $\lim_{x \rightarrow c} [f(x)g(x)] = \infty, L > 0, \infty \times L \quad +L$

$\lim_{x \rightarrow c} [f(x)g(x)] = -\infty, L < 0, \infty \times L \quad -L$

3. Quotient:  $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0 \quad \frac{L}{\infty} \quad \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \infty \quad \frac{\infty}{L}$

Similar properties hold for one-sided limits and for functions for which the limit of  $f(x)$  as  $x$  approaches  $c$  is  $-\infty$ .

$$\bullet \lim_{x \rightarrow 1^+} \left[ \frac{\sin x}{\frac{1}{x-1}} \right] = \lim_{x \rightarrow 1^+} \frac{\sin x}{\frac{1}{x-1}} = \# = 0$$

① Use quotient property

② Solve each limit  
separately. Use the  
TYPE of limit to evaluate.

\* by  
quotient  
property \*

$$\therefore \lim_{x \rightarrow 1^+} \left[ \frac{\sin x}{\frac{1}{x-1}} \right] = 0$$

③ Write ∴  
Statement

If limit @ a point = value of point then the continuity of the function at this point exists.

## CONTINUITY

### (4a) Definition & conditions for continuity

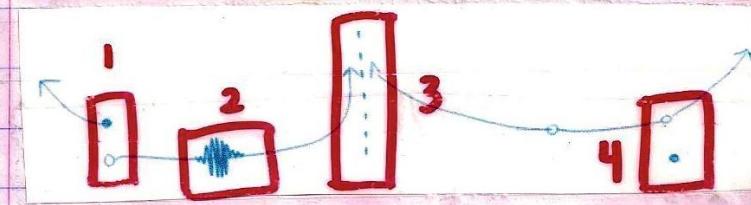
continuity: graph is continuous (unbroken) for its entire domain at every point.

#### conditions for continuity

$$\lim_{x \rightarrow a} f(x) = L$$

- $f(a)$  must be a defined value
- limit must exist (same value from both sides) as  $x \rightarrow a$
- $\lim_{x \rightarrow a} f(x) = f(a)$  ← ONLY IF ABOVE TWO CONDITIONS ARE SATISFIED

#### Types of discontinuities that can exist



conditions broken  
↓  
② ③

#### 1. JUMP DISCONTINUITY

This feature is similar to #4 but a jump has a point with a piece of the graph and a "hole" with a piece of the graph.

#### 2. OSCILLATING DISCONTINUITY

Infinity goes up and down and is undefined at one value of  $x$ .

#### 3. ESSENTIAL DISCONTINUITY (VA)

#### 4. REMOVABLE DISCONTINUITY (HOLE)

This feature is unlike #1 because the "hole" is part of the graph with an additional defined point elsewhere.

### (4b) CONTINUITY vs. the existence of a limit

Finding the limit at a certain point should not be confused with the principle of continuity. It is possible for a limit to exist even if the function is not continuous. AS LONG AS the graph approaches the same value,  $L$ , at the  $x$ -value  $a$  from both sides, the limit exists.

overall limit {  
LIMIT CAN EXIST → removable discontinuity is present  
LIMIT CANNOT EXIST → jump, oscillating, essential discontinuity is present

★ ★ Continuity talks about whether a graph is flowing continuously for all points/values in its domain. Existence of a limit at a certain point describes the behaviour of the graph at the value of  $a$ . ★ ★

(4)(c) Verifying Continuity / Identifying Continuity for NON-PIECEWISE Functions  
 TO LOOK AT:

- denominator  $\neq 0$
- radicand  $\geq 0$
- log input  $> 0$

Find values where function is continuous.

$$\bullet h(x) = \frac{x^2 + 16}{x^2 - 5x} = \frac{x^2 + 16}{x(x-5)}$$

$$\boxed{\begin{array}{l} \therefore x \neq 0 \\ x \neq 5 \end{array}}$$

① Notice this function has a denom. To identify values that would make the denominator  $= 0$ , find VAs. (Make each factor  $\neq 0$ )

$\therefore h(x)$  is continuous for  $x \in \mathbb{R}$   
 $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$

② Write  $\therefore$  stating when function is continuous

$$\bullet f(x) = \frac{\sqrt{x^2 - x - 6}}{x - 3}$$

denom  $\neq 0$   
 $x - 3 \neq 0$   
 $\boxed{x \neq 3}$

$x^2 - x - 6 > 0$   
 $(x+2)(x-3) > 0$

( -2      3 )

EVEN THOUGH we include 3 when testing intervals, we know that  $x \neq 3$

① Notice that you have a radicand and denominator to worry about.

Do radicand  $\geq 0$  and denom  $\neq 0$

② Solve the quadratic inequality

③ Write  $\therefore$  statement of when  $f(x)$  is continuous

$\therefore f(x)$  is continuous for  
 $x \in (-\infty, -2] \cup (3, \infty)$

\* If given a linear/quadratic log input, use an inequality (as you would have above) to solve for when function is continuous.

(4)(d) Verifying Continuity / Identifying Continuity for PIECEWISE functions

TO LOOK AT:

- verify that each piece is continuous
- look at where pieces disconnect/connect

We look at the continuity of functions themselves to identify other discontinuities aside from ones at special values.

$$\bullet f(x) = \begin{cases} \frac{1}{x-1} & x < 1 \\ x^3 - 2x + 5 & x \geq 1 \end{cases}$$

piece #1

$\frac{1}{x-1}$  is not continuous because of the VA at  $x=1$ . But, our domain for this piece is  $x < 1$  so, the rational graph that is less than 1 is continuous.

piece #2

continuous because it's a polynomial

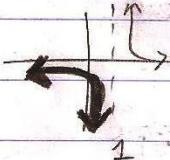
TO DETERMINE IF FUNCTION IS CONTINUOUS, USE ONE-SIDED LIMITS!

piece #1  $\lim_{x \rightarrow 1^-} \frac{1}{x-1} = \frac{1}{1-1} = \frac{1}{0} \therefore \text{undefined}$

(2) Now, we apply the conditions for continuity by looking at our special values  
(disconnected/connected points)

↳ be specific using a graph

$$\therefore \lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$



piece #2  $\lim_{x \rightarrow 1^+} x^3 - 2x + 5 = (1)^3 - 2(1) + 5 = 1 - 2 + 5 = 4 \therefore \lim_{x \rightarrow 1^+} x^3 - 2x + 5 = 4$

f(x)  $\therefore \lim_{x \rightarrow 1} f(x) = D, N, E$

(3) State the type(s) of discontinuity present

If the one-sided limits matched, the function would be continuous and this would be true:  $\lim_{x \rightarrow a} f(x) = f(a)$

The function is continuous for  $x \in (-\infty, 1) \cup (1, \infty)$  with an essential discontinuity at  $x=1$ .

↳ WHEN STATING WHERE IT IS CONTINUOUS OVERALL:

look at original domain restrictions and make note of other discontinuities.  
(holes, VAs, etc.)

(e) How to find missing constants to make a piecewise function continuous

$$f(x) = \begin{cases} -2x+a & x \leq -1 \\ x^2+b & -1 < x \leq 2 \\ \frac{1}{x}+2a & x > 2 \end{cases}$$

piece #1

linear  $\therefore$  continuous

piece #2

quadratic  $\therefore$  continuous

piece #3

denom  $\neq 0$

$x \neq 0$

① Verify that each piece is continuous

BUT,  $x=0$  is NOT in our domain so this piece is continuous

$x = -1$

$$\lim_{x \rightarrow -1^-} -2x+a = \lim_{x \rightarrow -1^+} x^2+b$$

(DIRECT SUB)

$$-2(-1)+a = (-1)^2+b$$

$$2+a = 1+b$$

$$\boxed{a=b-1}$$

TRY WORKING WITH

$x=2$  to be able  
to sub in.

$$a=b-1$$

$$a=\frac{11}{2}-1$$

$$\boxed{a=\frac{9}{2}}$$

$x = 2$

$$\lim_{x \rightarrow 2^-} x^2+b = \lim_{x \rightarrow 2^+} \frac{1}{x}+2a$$

(DIRECT SUB)

$$(2)^2+b = \frac{1}{2}+2a$$

$$4+b = \frac{1}{2}+2a$$

$$4+b = \frac{1}{2}+2b-2$$

$$4+2-\frac{1}{2}=b$$

$$\boxed{\frac{11}{2}=b}$$

USE TO FIND a

② Identify the disconnect/connect points

③ Using one-sided limits, evaluate values that can be used to find constant values,

REMEMBER:  
you want to make the one-sided limits equal to one another!

④ Write a statement

$\therefore$  To make  $f(x)$  continuous,  $a=\frac{9}{2}$ ,  $b=\frac{11}{2}$