

$$f'(x) = f \text{ prime of } x$$

⑧(a) The Derivative Function

The derivative function represents the slope of a curve at any point, at a general point, x . It is the expression that represents the slope of any point; it is not a specific value. (It is used to find specific slope values at specific points)

Notation: If $y = f(x)$, then the derivative is written as:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

and defined as a function, called the derivative of f , as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Direct definition

$$f'(x) = \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x}$$

Alternate definition

This is what you must use whenever you are asked to prove what the derivative is by definition or by first principles.

differentiable
⇒ derivative exists

A function f is differentiable at a if $f'(a)$ exists.

It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.

⑧(b) How to find the derivative by FIRST PRINCIPLES

- Find the slope of the tangent to $y = \sqrt{x-2}$ at $x=6$

DIRECT DEFINITION METHOD

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h}$$

RATIONALIZE!

* Realize that slope of tangent at a certain point = incl. But, we don't plug in until the end.

KEEP/USE BRACKETS

$$= \lim_{h \rightarrow 0} \frac{(x+h-2) - (x-2)}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \lim_{h \rightarrow 0} \frac{x+h-2-x+2}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \frac{1}{\sqrt{x-2} + \sqrt{x-2}}$$

THIS IS THE

$f'(x) = \text{DERIVATIVE OF}$

$f(x)$.

$$= \frac{1}{2\sqrt{x-2}}$$

① Identify definition

② Sub-in function

③ Notice that since you would get $\frac{0}{0}$ from D.S., manipulate/change form of limit.

④ Sub-in h when possible

⑤ Using the derivative, find slope at $x=6$.

⑥ Write statement

$$m_{\tan} = \frac{1}{2\sqrt{x-2}} = \frac{1}{2\sqrt{6-2}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

∴ Slope of tangent line of $f(x) = \frac{1}{4}$.

(by direct definition)

order of a/x
can be switched
but we say
 $\Rightarrow x$ we will be
left with x 's in the
derivative which is
what we want.

ALTERNATE DEFINITION METHOD

$$\begin{aligned} \frac{dy}{dx} &= \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x} = \lim_{a \rightarrow x} \frac{\sqrt{a-2} - \sqrt{x-2}}{a - x} \frac{(\sqrt{a-2} + \sqrt{x-2})}{(\sqrt{a-2} + \sqrt{x-2})} \quad (1) \text{ Identify definition} \\ &= \lim_{a \rightarrow x} \frac{(a-2) - (x-2)}{(a-x)(\sqrt{a-2} + \sqrt{x-2})} = \lim_{a \rightarrow x} \frac{a-2-x+2}{(a-x)(\sqrt{a-2} + \sqrt{x-2})} \quad (2) \text{ Sub-in function} \\ &= \lim_{a \rightarrow x} \frac{(a-x)}{(a-x)(\sqrt{a-2} + \sqrt{x-2})} = \lim_{a \rightarrow x} \frac{1}{\sqrt{a-2} + \sqrt{x-2}} \quad (3) \text{ Get rid of } \frac{0}{0} \text{ form} \\ &\qquad \qquad \qquad \text{by rationalizing} \\ &\qquad \qquad \qquad (4) \text{ Sub-in } a=x \\ &\qquad \qquad \qquad \text{when possible} \\ &= \lim_{a \rightarrow x} \frac{1}{\sqrt{a-2} + \sqrt{x-2}} = \frac{1}{\sqrt{x-2} + \sqrt{x-2}} = \frac{1}{2\sqrt{x-2}} \quad (5) \text{ Use derivative} \\ m_{tan} &= \frac{1}{2\sqrt{x-2}} = \frac{1}{2\sqrt{6-2}} = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}} \quad (6) \text{ Int. statement} \\ &= f'(x) \end{aligned}$$

\therefore Slope of tangent of $f(x)$ at $x=6$

$$is \frac{1}{4}.$$

(8) Sketching derivative graphs

THINGS TO NOTE:

→ Derivative of a polynomial will always be one degree less.

$$\begin{aligned} \text{ex. inc of } 10x^4 &= 10x^3 \\ &= (4)(10)x^3 \\ &= 40x^3 \end{aligned}$$

→ Sharp points have no derivative because $m=\text{undefined}$. So, there are holes or VAs there in the derivative graph THERE IS NO VALUE AT THIS POINT.

→ HA of original (no matter where it is) creates an H at $y=0$ in the derivative graph

→ VA of original stays the same in the derivative graph

→ Max/min points have a zero slope so take the input of original and make output 0.

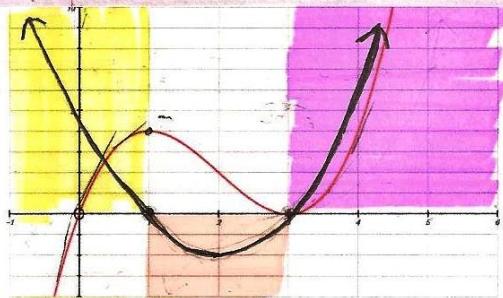
→ Cosine is the derivative of sine.

→ When sketching, draw end behaviour arrows and connect to points/holes/

I'm a donut
I donut know how
to do these very well...

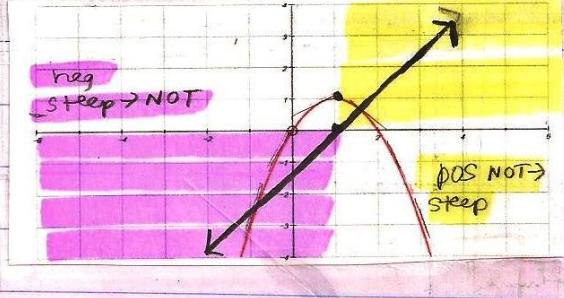
SKETCHING DERIVATIVE GRAPHS

• Polynomial Ex



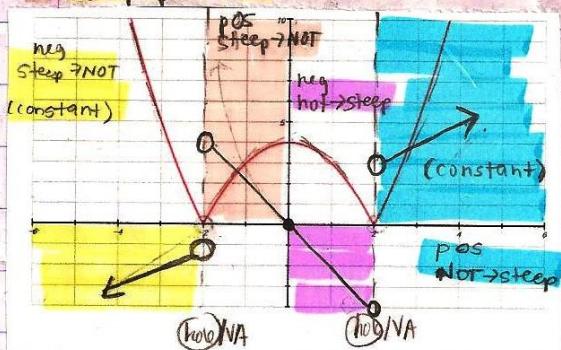
pos : steep \rightarrow NOT
neg : NOT \rightarrow STEEP \rightarrow NOT
pos : NOT \rightarrow steep

NOTE : x^3
 $y \frac{dy}{dx} = x^2$



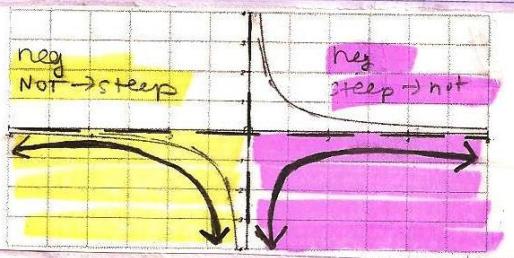
pos NOT \rightarrow steep

• Sharp points

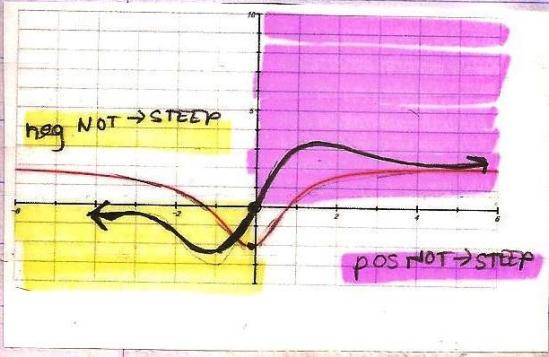


(no)VA

• HAVVA

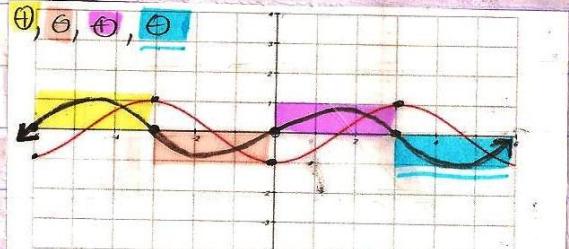


• Other



neg NOT \rightarrow STEEP

pos NOT \rightarrow STEEP



⊕, ⊖, ⊕, ⊖

note on sharp points: If $f(x)$ has a sharp point at $x=q$,
then there can be NO value of $f'(x)$ at $x=q$ ($q = \text{constant}/\text{dummy variable}$)

Proof of the Power Rule (General Version)

For this proof we'll again need to restrict n to be a positive integer. In this case if we define $f(x) = x^n$ we know from the alternate limit form of the definition of the derivative that the derivative $f'(x)$ is given by,

$$f'(x) = \lim_{\alpha \rightarrow x} \frac{f(x) - f(\alpha)}{x - \alpha} = \lim_{\alpha \rightarrow x} \frac{x^n - \alpha^n}{x - \alpha}$$

Now we have the following formula,

$$x^n - \alpha^n = (x - \alpha)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-3}x^2 + a^{n-2}x + a^{n-1})$$

You can verify this if you'd like by simply multiplying the two factors together. Also, notice that there are a total of n terms in the second factor (this will be important in a bit).

If we plug this into the formula for the derivative we see that we can cancel the $x - \alpha$ and then compute the limit.

$$\begin{aligned} f'(x) &= \lim_{\alpha \rightarrow x} \frac{(x - \alpha)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-3}x^2 + a^{n-2}x + a^{n-1})}{x - \alpha} \\ &= \lim_{\alpha \rightarrow x} x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-3}x^2 + a^{n-2}x + a^{n-1} \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \dots + a^{n-3}a^2 + a^{n-2}a + a^{n-1} \quad \leftarrow \text{all exponents on each } a \text{ become } (n-1) \\ &= n x^{n-1} \quad \text{since all terms are like terms, we multiply by } n \end{aligned}$$

After combining the exponents in each term we can see that we get the same term. So, then recalling that there are n terms in second factor we can see that we get what we claimed it would be.

To completely finish this off we simply replace the α with an x to get,

$$f'(x) = nx^{n-1}$$

Since there are
n # of terms..

① Finding the derivative using shortcut rules

• $y = 12x^{4/3}$

$$\frac{d}{dx} 12x^{4/3} = 12 \left[\frac{d}{dx} x^{4/3} \right] = 12 \left[\frac{4}{3} x^{(4/3)-1} \right]$$

$$= 12 \left[\frac{4}{3} x^{(1/3)} \right] = \boxed{16x^{1/3}}$$

① If finding the derivative, begin with correct notation.

$$\therefore \frac{d}{dx} 12x^{4/3} = 16x^{1/3}$$

② Notice there is a constant attached to the power function $x^{4/3}$.

So, use the constant multiple rule.

③ Within the brackets, use the power rule to evaluate. Then simplify constants

④ Write \therefore statement

• $y = x^2 + x^3$

$$\begin{aligned} \frac{d}{dx} x^2 + x^3 &= \frac{d}{dx} x^2 + \frac{d}{dx} x^3 = 2x^{2-1} + 3x^{3-1} \\ &= 2x + 3x^2 \end{aligned}$$

③ Use power rule

$$\therefore \frac{d}{dx} x^2 + x^3 = 2x + 3x^2$$

① Start by writing notation of finding the derivative.

② Notice there are 2 functions so, find the derivative of each. Use sum/difference rule.

④ Write \therefore statement

• $2x^4 + 3x^{5/3} - 4 = y$ (Write as $4x^{-1}$ to help)

SAME STEPS AS ABOVE

$$\frac{dy}{dx} = 2(4)x^{4-1} + 3\left(\frac{5}{3}\right)x^{(5/3)-1} - 4(-1)x^{(-1)-1}$$

$$\therefore \frac{dy}{dx} = 8x^3 + 5x^{2/3} + 4x^{-2} \quad \text{or} \quad 8x^3 + 5x^{2/3} + \frac{4}{x^2}$$

0a) Derivative Rules

Derivative of a Constant Function	$\frac{d}{dx}(c)=0$
The Power Rule (General Version) If n is any real number, then	$\frac{d}{dx}(x^n)=nx^{n-1}$
The Constant Multiple Rule If c is a constant and f is a differentiable function, then	$\frac{d}{dx}(c \cdot f(x))=c \cdot \frac{d}{dx}f(x)$
The Sum/Difference Rule If f and g are both differentiable, then	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

0b) Proof of sum/difference rule

(using direct definition)

SUM:

$$\left\{ \frac{d}{dx}[f(x) + g(x)] \right\} = \lim_{h \rightarrow 0} f(x+h) + g(x+h) - [f(x) + g(x)] \quad \text{(1) Sub-in using definition}$$

$$= \lim_{h \rightarrow 0} f(x+h) - f(x) + g(x+h) - g(x) \quad \text{(2) Split numerator}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \left(\frac{d}{dx}f(x) \right) + \left(\frac{d}{dx}g(x) \right)$$

DIFFERENCE: to prove this, follow steps
as shown above,

Proof of constant multiple rule

(using alternate definition)

$$\left\{ \frac{d}{dx}[c f(x)] \right\} = \lim_{a \rightarrow x} c f(a) - c f(x) \quad \begin{array}{l} \text{(1) Sub-in using definition} \\ \text{(2) common factor out } c \end{array}$$

$$= c \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x} = c \left[\frac{d}{dx}f(x) \right]$$

Proof of derivative of a constant function rule

(using direct definition)

$$\left\{ \frac{d}{dx}c = 0 \right\} \quad \frac{d}{dx}c = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} \quad \begin{array}{l} \text{(1) Identify definition} \\ \text{(2) Sub-in} \end{array}$$

$$= \lim_{h \rightarrow 0} 0 = 0$$

EXTRA: Leibniz notation

- $f = \frac{a^3 x^2}{2}$ $g = x^2$ $p = t^3 + 2t$

If other variables are present when using this notation, only derive independent variable. All others are constants.

$\frac{df}{dx} = \frac{\text{derivative of } f \text{ with respect to } x}{}$

$$\frac{df}{dx} = \frac{2x a^3}{2} \leftarrow \text{ONLY USE POWER RULE ON } x$$

$$\frac{df}{da} = \frac{3a^2 x^2}{2} \leftarrow \text{ONLY USE POWER RULE ON } a$$

$$\frac{dp}{dt} = 3t^2 + 2 \leftarrow \text{ONLY USE POWER RULE ON } t \text{ (only variable here)}$$

$$\frac{dg}{dx} = 2x \quad \frac{dg}{dy} = 0 \leftarrow \text{NB independent variable "y".}$$

NOTES

- power rule on linear variable produces constant.
- power rule on constant produces zero.

$$3x^0 = 3(0)x^{-1} = 0$$

- derivative of speed = acceleration!
(velocity)

②(a) Definition and conditions for differentiability

Differentiability \rightarrow function, at a certain point, is differentiable if the derivative exists.

Conditions for differentiability

A function f is differentiable at a number a if it is continuous and smooth at a . THEREFORE (mathematically) ...

continuous \rightarrow ① $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a) \rightarrow$ continuous (defined) at a

smooth \rightarrow ② $\lim_{x \rightarrow a^-} f'(x) = \lim_{x \rightarrow a^+} f'(x) \rightarrow$ derivatives/slopes from both sides match

★ a must be defined because the derivative at a can only exist if it is a defined point. ★

not continuous then not smooth

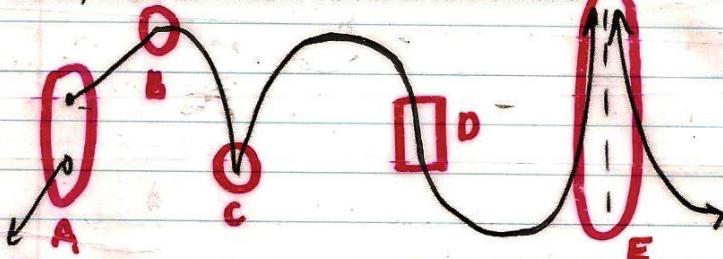
Types of non-differentiability

(PSSST! WANNA CHEAT?)



- piecewise - jump/cusp
- absolute value - corner

Identify where the function will not be differentiable.



- odd even
- cusp
- odd even
- vertical tangent
- log, rational, some trig - VA

- Jump discontinuity → condition ① for differentiability is broken since one-sided limits can't be the same. AUTOMATICALLY, smoothness condition cannot be satisfied.
- Corner → condition ① ✓ but condition ② is broken because the slope on either side of the point will NOT be the same so $f'(x)$ of both sides won't match.
- Vertical Tangent → condition ① ✓ but condition ② is broken because the derivative (slope) of a vertical line is undefined and DNE.
- Essential discontinuity → condition ① is broken because the pt $f(a)$ is not defined. A VA is a discontinuity that also creates an undefined value. The derivative cannot be taken at an undefined value. This is why condition ② is broken.

②(b) **Theorem:** If f is differentiable at a , then f is continuous at a

Is the converse (FLIPPED) true?

If f is continuous at a , is f differentiable at a ? NO, not always!

The converse is saying that, automatically, if f is defined at a , f is differentiable at a — which cannot be stated until we prove it.

$\lim_{x \rightarrow a^-} f'(x) = \lim_{x \rightarrow a^+} f'(x)$. Even if f is defined at a , there

could be a corner or cusp at the value of a (making it indifferentiable).

(using alternate definition)

① Proof using 1st principles, that a function IS NOT differentiable at a particular point (and differentiable at another)

$$\bullet f(x) = \begin{cases} x+3, & x < -2 \\ (x+3)^2, & x \geq -2 \end{cases}$$

Prove that $f(x)$ is not differentiable at $x = -2$.

continuity?

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^+} f(x) = f(-2)$$

$$\lim_{x \rightarrow -2^-} x+3 = \lim_{x \rightarrow -2^+} (x+3)^2 = (-2+3)^2$$

DIRECT SUB.

$$-2+3 = (-2+3)^2 = (-2+3)^2$$

smoothness?

$$1 = 1 = 1 \therefore f(x) \text{ is continuous}$$

$$f'(x) = \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x}$$

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(-2)}{x - (-2)}$$

LEFT:

$$f'(x) = \lim_{x \rightarrow -2^-} \frac{(x+3) - f(-2)}{x + 2}$$

$$f'(x) = \lim_{x \rightarrow -2^-} \frac{x+3 - 1}{x+2}$$

$$f'(x) = \lim_{x \rightarrow -2^-} \frac{x+2}{x+2} = 1 \quad (1)$$

RIGHT:

$$f'(x) = \lim_{x \rightarrow -2^+} \frac{(x+3)^2 - f(-2)}{x+2}$$

$$f'(x) = \lim_{x \rightarrow -2^+} \frac{x^2 + 6x + 9 - 1}{x+2}$$

$$f'(x) = \lim_{x \rightarrow -2^+} \frac{(x+2)(x+4)}{(x+2)}$$

$$f'(x) = \lim_{x \rightarrow -2^+} (x+4) = -2+4 = 2 \quad (2)$$

ABOUT ALTERNATE:
since we want
to look at
 $x \rightarrow -2$ (deriv. @
 $x = -2$) we can
change $a \rightarrow x$
to $x \rightarrow -2$

then modify the
definition to match!

① Verify if the first
condition (continuity)
for differentiability
is true.

since one-sided
limits match $f(-2)$
and $f(-2)$ is
defined, we can
begin to test
smoothness.

② Find the derivative at
 $x = -2$ using alternate
definition on the
LEFT and RIGHT side.

③ Write \therefore statement

Compare
the derivatives
(slopes) of
the LS and RS
of the point $x = -2$.
If same, then
differentiable.
If not, then not.

$$\lim_{x \rightarrow -2^-} f'(x) \neq \lim_{x \rightarrow -2^+} f'(x)$$

$$1 \neq 2$$

\therefore at $x = -2$, $f(x)$ is not
differentiable

OH NO!
WE FORGOT
TO STATE
THAT AT $x = -1$,
 $f(x)$ IS CONTINUOUS.
TO PROVE THIS:
 $\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} f(x)$
 $4 = 4$ AND $f(-1) = 4$

DON'T
FORGET
THIS!
...like I just
(almost) did.

- for the same function, prove $f(x)$ is differentiable at $x = -1$

$$f'(x) = \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x}$$

change! we want $x \rightarrow -1$

$$f'(x) = \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{(x+3)^2 - (-1+3)^2}{x+1}$$

$$f'(x) = \lim_{x \rightarrow -1} \frac{x^2 + 6x + 9 - 4}{x+1}$$

$$f'(x) = \lim_{x \rightarrow -1} \frac{(x+3)(x+5)}{(x+1)}$$

$$f'(x) = \lim_{x \rightarrow -1} \frac{(x+3)(x+5)}{(x+1)}$$

$$f'(x) = \lim_{x \rightarrow -1} (x+5) = (1+5) = 4$$

\therefore since $\lim_{x \rightarrow -1} f'(x) = 4$, the function

$f(x)$ is differentiable at $x = -1$.

① Identify definition
that will be used

② Sub-in $f(x)$.

NOTE: since we are
doing limit as $x \rightarrow -1$,
only use piece #2.
(domain is $\therefore x \geq -2$)

also use this piece
to evaluate $f(-1)$

③ Since $f'(x)$

at $x = -1$ exists,
then the
function is
differentiable at $x = -1$

④ Write statement

EXTRA! Identifying types of non-differentiabilities

What type of non-differentiability does $y = \sqrt[3]{x}$ have?

① Verify that it
is continuous.

Domain of $\sqrt[3]{x}$ is $x \in \mathbb{R} \therefore$ it is continuous.

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\frac{d}{dx} x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}} = \left(\frac{1}{3} x^{-\frac{2}{3}}\right)$$

$$y' \Rightarrow \lim_{x \rightarrow 0^-} \frac{1}{3} x^{-\frac{2}{3}} = \infty \quad (\text{try } \#s)$$

$$y' \Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{3} x^{-\frac{2}{3}} = \infty \quad (\text{try } \#s)$$

using list, since the limit is approaching
the same infinity, we can conclude that:

$y = \sqrt[3]{x}$ has a vertical tangent
and is not differentiable at $x = 0$.

② Find the
derivative and
using the point
of interest $(0,0)$
see if the slopes
from both sides
match, etc.

③ Based on answers,
since the limit DNE,
at $x = 0$, the function
is non-differentiable.

④ To determine type,
refer to list on next pg.

NOTE: List can only be used to aid in conclusions of algebraic proof for non-differentiability.

EXTRA!

LIST OF: characteristics to identify types of non-differentiable features (of a function's graph)

If: f has a VA, f' will have a VA.

If: f has a vertical tangent, f' will have a VA where limit approaches the same infinity. ($\pm\infty$)

If: f has a cusp, f' will have a VA where limit approaches opposite infinities (+ or - ∞)

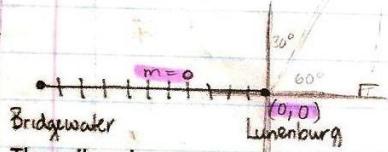
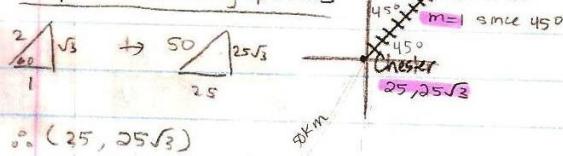
If: f has a corner or jump, f' will have a jump. (+ hole)

EXTRA!

Application of differentiability \rightarrow CONNECTIVITY

The Clickety-Clack Railroad Company has laid out a track from Bridgewater to Lunenburg and also has a track in operation from Chester to Hubbards, as shown. Place origin at Lunenburg.

Find pt chester using special Δ



The railroad company wishes to connect the path smoothly. Assume that Lunenburg is due east of Bridgewater, Chester is 50 km from Lunenburg in a N30°E direction, and Hubbards is located N45° from Chester.

Create conditions

- (1) at $x=0$, $m=0$.
- (2) need pt $(0, 0)$
- (3) need pt $(25, 25\sqrt{3})$
- (4) at $x=25$, $m=1$

now, start plugging this in.

since 4 conditions, we are looking

for a CUBIC. $y = ax^3 + bx^2 + cx + d$
 $(y' = 3ax^2 + 2bx + c)$

$$\lim_{x \rightarrow 0^-} y' = \lim_{x \rightarrow 0^+} y' \quad \lim_{x \rightarrow 0^-} 0 = \lim_{x \rightarrow 0^+} 0$$

$$0 = 3a(0)^2 + b(0) + c \quad 0 = c$$

$$\text{use } c=0 \text{ now}$$

$$0 = a(0)^3 + b(0)^2 + d \quad 0 = d$$

$$\text{use } d=0 \text{ now}$$

$$\text{sub. pt } 25, 25\sqrt{3}$$

$$\frac{3\sqrt{3}}{25} = a(25)^3 + b(25)^2 \quad \text{plug in then divide by 25}$$

$$\sqrt{3} = 625a + 25b$$

$$\lim_{x \rightarrow 25^-} y' = \lim_{x \rightarrow 25^+} y'$$

$$\lim_{x \rightarrow 25^-} 3ax^2 + 2bx = 1$$

$$3a(25)^2 + 2b(25) = 1$$

$$1875a + 50b = 1$$

do elimination with these

$$1875a + 50b = 1 \quad \text{cancel}$$

$$2(625)a + 2(25)b = 2(\sqrt{3}) \quad \text{cancel}$$

$$\frac{625a}{625} = \frac{1-2\sqrt{3}}{625} \quad \text{FIND } B$$

$$a = \frac{1-2\sqrt{3}}{625} \approx -0.0039$$

$$\frac{1-1875a}{50} = b$$

$$0.1678 \approx b$$

$$\therefore Y = -0.0039x^3 + 0.1678x^2$$

use with...
COMPOSITIONS!

(3a) CHAIN RULE

The Chain Rule: If f and g are both differentiable and $F=f \circ g$ is the composite function defined by $F(x)=f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x)=f'(g(x))g'(x)$$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

"take the derivative of the outside function, and keep the inside the same. Multiply this with the derivative of the inside function"

(3b) Using The chain rule

$y=f(g(x))$	$u=g(x)$	$y=f(u)$	$\frac{dy}{du} \frac{du}{dx}$ in terms of u and x	$\frac{dy}{dx}$ in terms of x only
i. $y=(6x-5)^4$	$u=6x-5$	$y=u^4$	$(4u^3)(6)=24u^3$	$24(6x-5)^3$
ii. $y=\sqrt{x^2-1}$	$u=x^2-1$	$y=u^{1/2}$	$(\frac{1}{2}u^{-1/2})(2x)$	$x(x^2-1)^{-1/2}$

• $y=(6x-5)^4$

$$y=f(u) \quad f=u^4 \quad g=6x-5$$

$u=g(x) \rightarrow$ inside function $\rightarrow u=6x-5$

$y=f(u) \rightarrow$ outside function $\rightarrow y=u^4$

① Identify, using a letter u , the equations you are working with

Right now, our output is y ✓ but our input is u . ✗

We need to use the chain rule to "link" the functions so that we only have x 's left to differentiate.

② We are asked to find the derivative of y in terms of x .

so: $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

} Here, we are saying that derivative of y = derivative of f multiplied by the derivative of g (that is, derivative of y in terms of u multiplied by derivative of u in terms of x)

* y is the outside function; u is the inside function.

we are doing the derivative of outside + keeping what's inside then multiplying it with derivative of the inside.

*Use chain rule until the derivative ONLY IS 1! *

$$y = u^4 \quad u = 6x - 5$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

) when going from this line
to next, think of bringing exponent down then subtracting one.

(3) Differentiate

$$\frac{dy}{dx} = (4u^3)(6)$$

) simplify constants extended step: $(4u^{4-3})(1 \cdot 6x^{1-1} - 5 \cdot 0x^{0-1})$

in terms of $u \rightarrow \frac{dy}{dx} = 24u^3$ (4) Sub-in $= (4u^3)(6)$

in terms of $x \rightarrow \frac{dy}{dx} = 24(6x-5)^3$ Function u
(inside function) at this point. We want in terms of x but we have no x 's.

- $y = \sqrt{x^2-1}$

$$y = f(u) \quad f = u^{1/2} \quad g = x^2 - 1$$

$$u = g(x) \rightarrow \text{inside function} \rightarrow u = x^2 - 1$$

$$y = f(u) \rightarrow \text{outside function} \rightarrow y = u^{1/2}$$

Chain Rule says: take derivative of outside and keep inside same \times derivative of inside.

(1) Identify, using u , the equations you are working with.

(2) Since we are asked to find $\frac{dy}{dx}$, we need to use the chain rule.

in terms of $u \rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $y = u^{1/2} \quad u = x^2 - 1$

$$\frac{dy}{dx} = \left(\frac{1}{2}u^{-1/2}\right)(2x)$$

) THINK:
Bring exponent down
then subtract 1 for each

(3) Differentiate

$$\frac{dy}{dx} = \frac{1}{2}(x^2-1)^{-1/2}(2x)$$

) simplify constants (4) Sub-in function u

in terms of $x \rightarrow \frac{dy}{dx} = x(x^2-1)^{-1/2}$

Differentiate using chain rule in one step!

- $y = (6x-5)^4$

Take derivative of outside function by bringing down exponent and subtracting one while keeping inside same THEN multiply by derivative of inside.

$$\frac{dy}{dx} = 4(6x-5)^3(6)$$

) simplify constants

$$= 24(6x-5)^3$$

$$y = \sqrt{x^2-1} = (x^2-1)^{1/2}$$

Take derivative of outside function by bringing down exponent and subtracting one while keeping inside same THEN multiply by derivative of inside.

$$\frac{dy}{dx} = \frac{1}{2}(x^2-1)^{-1/2}(2x)$$

) simplify constants

$$= x(x^2-1)^{-1/2}$$

④ a) Product + Quotient Rules

The Product Rule:

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] = fg' + gf'$$

The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} = \frac{gf' - fg'}{g^2}$$

TALK TO
URSELF

Embrace your
inner nerd :)

Product Rule: (original) (deriv. of 2nd) + (original) (deriv. of 1st)

Quotient Rule: (bottom) (deriv. of top) - (top)(deriv. of bottom)

$$(bottom)^2$$

(3c)

Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \frac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

given

$$g(5) = -3$$

$$g'(5) = 6$$

$$h(5) = 3$$

$$h'(5) = -2$$

looking for

$$f'(5) \text{ in } abcd$$

a) $f(x) = g(x)h(x)$ ← To take the derivative of a product, use PRODUCT RULE

$$f'(x) = g(x)h'(x) + h(x)g'(x) \leftarrow \text{THIS IS THE DERIVATIVE. PLUG IN } x=5$$

$$f'(5) = g(5)h'(5) + h(5)g'(5)$$

$$= (-3)(-2) + (3)(6)$$

$$= 6 + 18 = 24$$

$$\therefore f'(5) = 24$$

b) $f(x) = g(h(x))$ ← THIS IS A COMPOSITION. SO, USE CHAIN RULE

$$f'(x) = g'(h(x)) \cdot h'(x) \quad \text{PLUG IN } x=5$$

$$f'(5) = g'(h(5)) \cdot h'(5)$$

$$= g'(3) \cdot (-2)$$

IT IS NOT POSSIBLE TO GO ON USING THE INFORMATION PROVIDED
BECAUSE $g'(3)$'s output is not given. We need it to continue.

c) $f(x) = \frac{g(x)}{h(x)}$ ← TO TAKE DERIVATIVE, USE QUOTIENT RULE.

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2} \quad \leftarrow \text{PLUG IN } x=5$$

$$f'(5) = \frac{h(5)g'(5) - g(5)h'(5)}{[h(5)]^2} = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{4}{3}$$

$$\therefore f'(5) = \frac{4}{3}$$

d) $f(x) = [g(x)]^3$ ← This is a composition. Use CHAIN RULE to differentiate.

$$f'(x) = 3[g(x)]^2 g'(x)$$

$$f'(5) = 3[g(5)]^2 g'(5)$$

$$= 3(-3)^2(6)$$

$$= 162$$

$$\therefore f'(5) = 162$$

If $y = u\sqrt{1-u}$ and $u = 3x - x^2$

$$\text{determine } \left. \frac{dy}{dx} \right|_{x=-1}$$

"derivative of y with respect
to x at $x = -1$."

$$y = u\sqrt{1-u}$$

← sub in for u

$$y' = fg' + gf' \quad y = (3x-x^2)(1-3x+x^2)^{1/2} \quad \leftarrow \text{differentiate using product rule & chain rule}$$

$$y' = (3x-x^2)^{\frac{1}{2}}(1-3x+x^2)^{-\frac{1}{2}}(-3+2x) + (1-3x+x^2)^{\frac{1}{2}}(3-2x)$$

$$\text{Factor} \quad y' = \frac{1}{2}(3x-x^2)(1-3x+x^2)^{-\frac{1}{2}}(3-2x) + (1+3x+x^2)^{\frac{1}{2}}(3-2x)$$

$$(\text{optional}) \quad y' = (3-2x)(1-3x+x^2)^{-\frac{1}{2}} \left[\left(-\frac{1}{2} \right)(3x-x^2) + (1+3x+x^2) \right]$$

Sub in

$$x = -1 \quad y' = (3-2(-1))(1-3(-1)+(-1)^2)^{-\frac{1}{2}} \left[\left(-\frac{1}{2} \right)(3(-1)-(-1)^2) + (1-3(-1)+(-1)^2) \right]$$

$$y' = (5)\left(\frac{1}{\sqrt{5}}\right)(2+5)$$

$$y' = \frac{35}{\sqrt{5}} = \frac{35\sqrt{5}}{5} = 7\sqrt{5}$$

$$\therefore \left. \frac{dy}{dx} \right|_{x=-1} = 7\sqrt{5}$$

(b) Proofs for product and quotient rule $f(x) = p(x)q(x)$

DIRECT DEF.

• PRODUCT RULE $f'(x) = p(x)q'(x) + q(x)p'(x)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{p(x+h)q(x+h) - p(x)q(x)}{h}$$

① Sub functions into definition

$$= \lim_{h \rightarrow 0} \frac{p(x+h)q(x+h) - p(x+h)q(x) + p(x+h)q(x) - p(x)q(x)}{h}$$

② Manipulate by subtracting and adding a product of first and last monomials.

$$= \lim_{h \rightarrow 0} \frac{p(x+h)[q(x+h) - q(x)] + q(x)[p(x+h) - p(x)]}{h}$$

so: $p(x+h)q(x)$

③ Common factor on TDP

$$= \lim_{h \rightarrow 0} p(x+h) \lim_{h \rightarrow 0} q(x+h) - q(x) + \lim_{h \rightarrow 0} q(x) \lim_{h \rightarrow 0} p(x+h) - p(x)$$

↑ ↑

DO DIRECT SUBS
OF $h=0$ $\rightarrow \lim_{h \rightarrow 0} p(x) = p(x)$ $\lim_{h \rightarrow 0} q(x) = q(x)$

④ Split expression
into 4 parts
using limit
properties.

$$f'(x) = p(x)q'(x) + q(x)p'(x) \quad \leftarrow \text{PRODUCT RULE}$$

• QUOTIENT RULE $f(x) = \frac{p(x)}{q(x)}$ $f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{q(x)^2}$

DIRECT DEF. ← RULE D

⑤ Use notes +
definition to
rewrite this
expression
using prime
notation.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{p(x+h)}{q(x+h)} - \frac{p(x)}{q(x)}}{h}$$

② Rewrite as $\frac{1}{h}$
and do LCM
on top

① Sub functions
into definition

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{h} p(x+h)q(x) - p(x)q(x+h)}{q(x+h)q(x)}$$

③ Manipulate by
subtracting and
adding product
of two original
functions. So:
 $p(x)q(x)$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{h} p(x+h)q(x) - p(x)q(x) + p(x)q(x) - p(x)q(x+h)}{q(x+h)q(x)}$$

$$= \lim_{h \rightarrow 0} \frac{1}{q(x+h)q(x)} \frac{p(x+h)q(x) - p(x)q(x) + p(x)q(x) - p(x)q(x+h)}{h}$$

④ Put larger
numerator over
denominator h .

$$= \lim_{h \rightarrow 0} \frac{1}{q(x+h)q(x)} \frac{q(x)[p(x+h) - p(x)] + p(x)[q(x+h) - q(x)]}{h}$$

⑤ Break up
large fraction @

$$= \lim_{h \rightarrow 0} \frac{1}{-q(x)q(x)} \left[\lim_{h \rightarrow 0} q(x) \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} - \lim_{h \rightarrow 0} p(x) \lim_{h \rightarrow 0} \frac{q(x+h) - q(x)}{h} \right]$$

addition sign and
common factor.

$$= \frac{1}{[q(x)]^2} [q(x)p'(x) - p(x)q'(x)]$$

⑥ Split using
properties of limits

$$f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2} \quad \leftarrow \text{QUOTIENT RULE!}$$

⑦ Write using prime
notation

*disclaimer: mistake was made in the process of this journal question
so be careful when using quotient rule!

(4c) Finding the derivative using product/quotient rules + simplifying

$$\bullet y = \left(\frac{\sqrt{1+3x}}{3x} - \frac{1}{\sqrt{1+3x}} \right) \quad \textcircled{1} \text{ LCD}$$

rearrangement
of quotient rule

$$y = \frac{(\sqrt{1+3x})^2 - 3x}{(3x)(\sqrt{1+3x})} = \frac{1}{(3x)/\sqrt{1+3x}} = \frac{1}{(3x)(1+3x)^{1/2}} \quad \textcircled{2} \text{ Apply quotient rule}$$

$$\downarrow y' = -f'g^{-1} + fg'^{-1} \quad y' = -(1) \frac{d}{dx}(3x)(1+3x)^{1/2} + (3x)(1+3x)^{1/2} \frac{d}{dx}(1)$$

$$y' = -[3x] \frac{1}{2} (1+3x)^{-1/2} (3) + (1+3x)^{1/2} (3) \cancel{+ (3x)} \cancel{(1+3x)^{1/2}}$$

$$y' = \frac{9x^2 (1+3x)}{2x(1+3x)^{1/2}} \quad \textcircled{3} \text{ Product rule must also be applied (+ chain rule within the product rule)}$$

$$y' = -\frac{3(1+3x)^{-1/2} [\frac{3}{2}x + 1+3x]}{9x^2 (1+3x)} = \frac{-(\frac{3}{2}x + 1+3x)}{3x^2 (1+3x)^{3/2}}$$

combine both
 $(1+3x)$ brackets

simplify 3 and 9

$$(1+3x)^{1-(-1/2)} = (1+3x)^{3/2}$$

$\textcircled{4} \text{ Simplify}$

EXTREMELY IMPORTANT NOTE: When using the quotient rule, NAME THINGS BECAUSE (as seen above) the subtraction must occur with the term fg' and if the two terms in the numerator are switched, the negative will not be distributed correctly. A quick fix (again, as seen above) is to put the negative as a factor around its correct term.

(f) (g)

• $y = (\sqrt[3]{x})(2x-x^2)$ $= x^{1/3}(2x-x^2)$ $y' = fg' + gf'$ (1) Rewrite using exponent form

$$y' = (x)^{1/3} \frac{d(2x-x^2)}{dx} + (2x-x^2) \frac{d(x)^{1/3}}{dx}$$

$\frac{d}{dx}$ $\frac{d}{dx}$

$= (x)^{1/3} (2-2x) + (2x-x^2) \frac{2}{3}(x)^{-1/3}$ ← FACTOR: (2) Apply product rule

$= 2x^{-1/3} \left[(x)(1-x) + \frac{1}{3}(2x-x^2) \right]$ (3) Simplify

$= 2(x-x^2 + \frac{2x}{3} - \frac{x^2}{3})$ ↗ EXPAND IN BRACKETS

$\frac{x^{1/3}}{x^{1/3}}$

$y' = 2(\frac{5}{3}x - \frac{4}{3}x^2)$

• $y = \frac{x^2+3}{x}$ (1) DON'T THINK QUOTIENT RULE RIGHTAWAY!
SIMPLIFY FIRST!

$$y = \frac{x^2}{x} + \frac{3}{x}$$

$$y = x + \frac{3}{x}$$

$$y = x + 3(x)^{-1}$$

(2) use power rule!

$$y' = 1 - 3(x)^{-2}$$

with respect to x

(7d) Examples of finding the derivative (Find $\frac{dy}{dx}$) (Find y')

• $y = 2^x$

$y' = (2^x)(\ln 2)$

$y' = (2^x)(\ln 2)$

① This is an exponential. To find the derivative,

think:

(original) (\ln of base) (derivative of exponent)

• $y = x^2$

$y' = 2x^{2-1}$

$y' = 2x$

① This is a polynomial. Apply power rule.

think:

multiply by exponent then subtract 1 from
the exponent.

• $y = e^{\pi}$

$y' = 0$

① There are no x 's to differentiate the function
in terms of. (DON'T FALL FOR MRS. K'S TRICKS)