

## Velocity and Acceleration

④(a) What do the variables  $s$ ,  $v$ ,  $a$  represent?

$s$  = displacement = position (distance BUT accounting for going forwards AND backwards over time)

$v$  = velocity (speed over time)

$a$  = acceleration (speeding up or down over time)

How are they related?

$s-t$   $f$  is the relationship between where the object moves over time.

$v-t$   $f'$  is the derivative of the  $s-t$  relationship. This tells how

$a-t$  fast/slow an object is moving over time. It also tells us WHERE,

$f''$  is the derivative of the  $v-t$  relationship. This tells how <sup>it's</sup> going. an object is speeding up/down over time.

**EXTRA:** To draw an  $s-t$  graph from a  $v-t$  graph, take into account that you don't know where/what the y-values are. We only know behaviour at certain/for certain x-values.

Relationships between displacement, velocity + acceleration  
in function/prime notation.

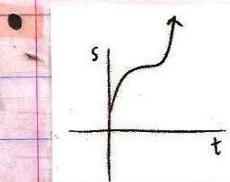
$$s'(t) = v(t)$$

$$(s''(t) = v'(t)) = a(t)$$

notes

negative acceleration  $\rightarrow$  decreasing velocity } much like velocity and  
positive acceleration  $\rightarrow$  increasing velocity } displacement

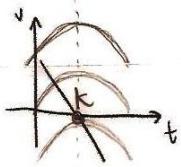
(4b) Why an increasing s-t graph may not necessarily mean speeding up



We need to remember that the y-values of an s-t graph represent displacement or position from a reference point. Here, the reference point is the origin. From this, the fact that the y-values have a positive slope tells us the object is moving north/right of the origin.

To analyze when an object is speeding up, we need to take the double derivative of s and use an a-t graph.  
(acceleration)

(4c) Why zero on a v-t graph does not mean an object is at the origin?



We need to consider the relationships between f and f'.

f has t.p or saddle  $\leftrightarrow$  f' has a zero

f has an inflection pt  $\leftrightarrow$  f' has a t.p

Since f' (v-t) graph has a zero, it means that f (s-t) graph will have a t.p or saddle pt somewhere along the x-value K where the zero occurs.

So, in fact, at the x-value K, there will be a t.p where the object changes its direction from North to South. It will be a t.p because the sign of v-t changes from (+) to (-) at K.

North      South

velocity  $\uparrow$  = NORTH (in original)  
 velocity  $\downarrow$  = SOUTH (in original)

(4b)

Condition	Event
$s \cdot v < 0$	Object moving toward the origin
$s \cdot v > 0$	Object moving away from the origin
$s \cdot a < 0$	Acceleration is directed toward the origin
$s \cdot a > 0$	Acceleration is directed away from the origin
$v \cdot a < 0$	Object is slowing down
$v \cdot a > 0$	Object is speeding up

Condition	Event
$s < 0$	Object to the left (below) of the origin
$s = 0$	Object at the origin
$s > 0$	Object to the right (above) of the origin
$v < 0$	Moving to the left (downward)
$v = 0$	At rest
$v > 0$	Moving to the right (upward)
$a < 0$	$s-t$ graph concave down start slow, speed up Acceleration directed to the left (downward)
$a = 0$	Constant velocity
$a > 0$	$s-t$ graph concave up slow down, speed up Acceleration directed to the right (upward)

(4e) Explain what is happening in an  $s-t$  graph and sketch its  $v-t$  graph

What is happening?

A: Object is stopped on the LS of the origin / South of the origin.

B: Speeding up towards the origin / north towards origin. (Since CU, it is accelerating)

C: Slowing down going away from the origin to the RS / going further north of the origin (Since CD, it is slowing down)

D: Constant velocity past the origin to the LS/south of the origin.

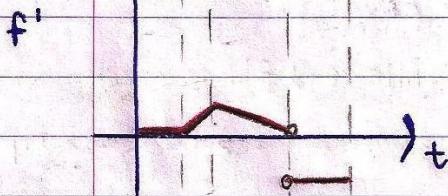
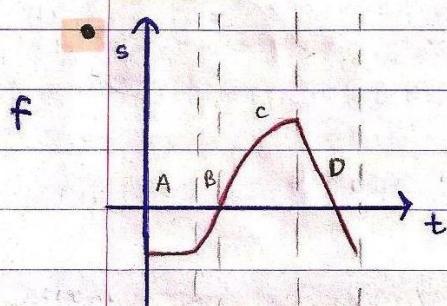
To sketch  $v-t$ :

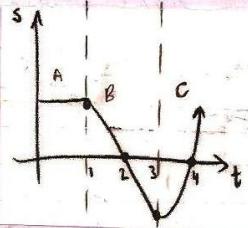
(1) All of (A) has slope of zero. So, this will be a flat line at zero.

(2) (B) starts out positive, almost flat, then ends positive, higher value.

(3) (C) starts out at highest value of (B) then slowly decreases speed to being positive but almost flat.

(4) (D) doesn't start at the same slope value (C) ended at. Rather, the negative slope value of (D) starts out at 1/s its constant slope.



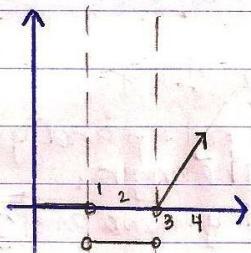


What is happening?

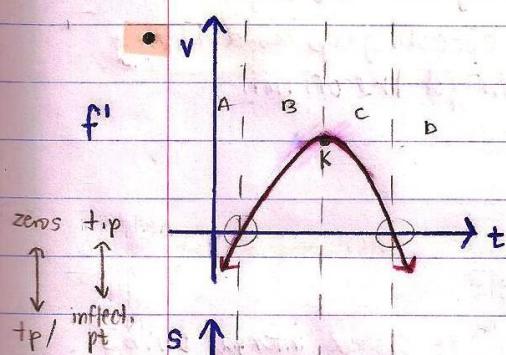
A: Object is stopped on the RS of the origin/north of the origin

B: Object has a constant velocity where it has turned and is going down towards then past the origin to the LS/is going South towards the origin then passing it.

C: Object is speeding up and moving towards the origin but continues to the RS/+ is speeding up and going north to the origin then passing it.

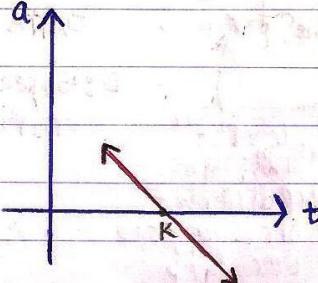


(4)(c) Explain what is happening in a v-t graph, sketch its s-t graph and a-t graph.



To sketch a-t:  
Treat v-t as f and draw f'.

so: t.p at unknown point, K, will become a zero.



What is happening?

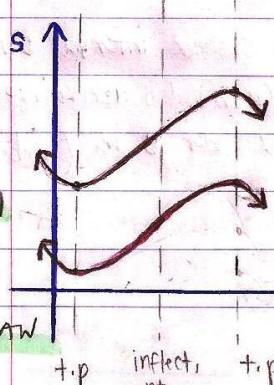
A: Object is slowing down and coming towards the origin/travelling South to the origin.

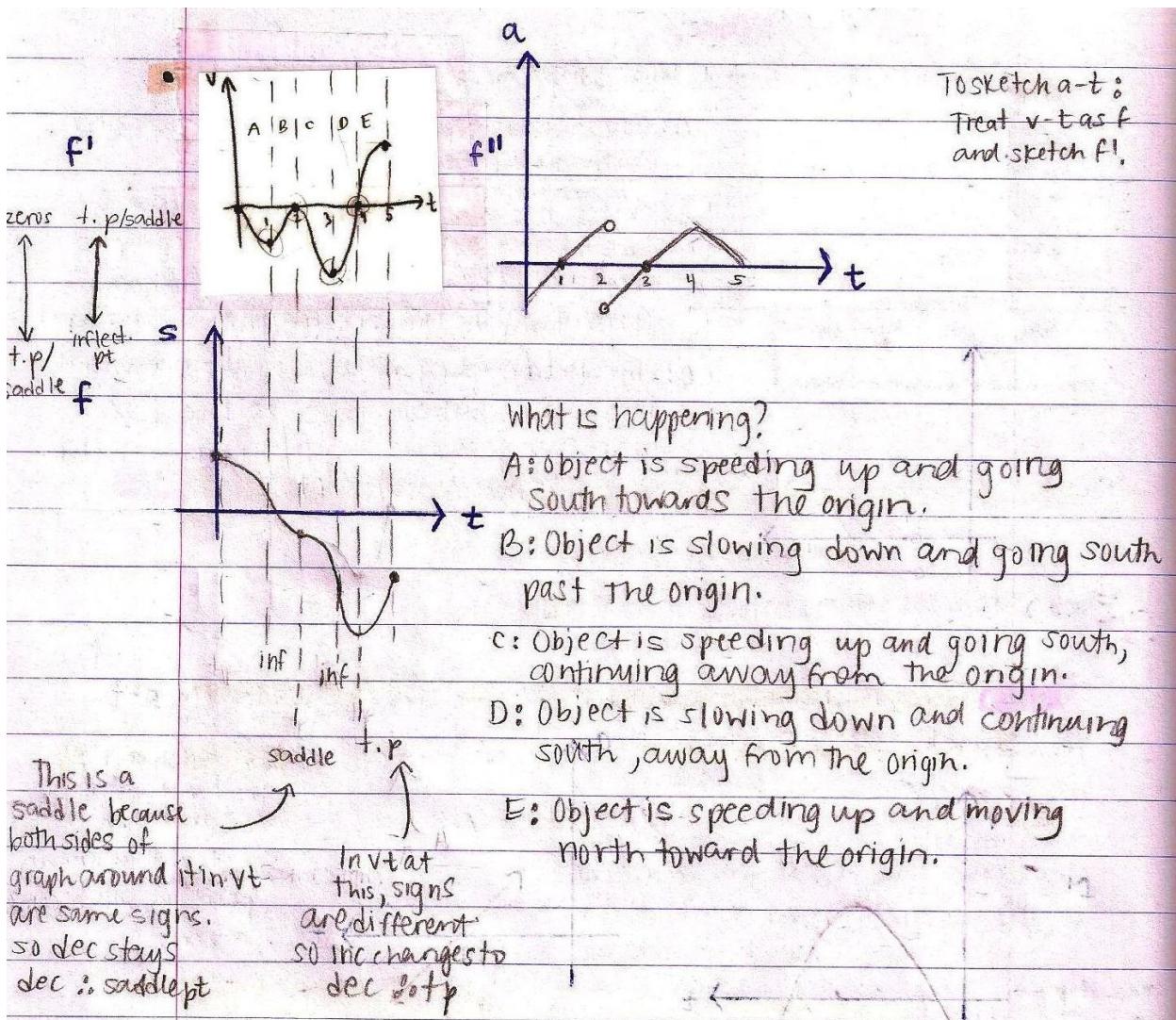
B: Object is speeding up and heading North/away from the origin.

C: Object is slowing down and still heading North/away from the origin.

D: Object is speeding up and heading South/towards the origin (then past the origin).

SINCE WE  
DON'T KNOW  
WHERE  
OBJECT  
STARTS, DRAW  
VARIOUS  
ST GRAPHS





- (a) What are local and absolute extreme max/min points?
- A local minimum point occurs when a point,  $c$ , has an output  $f(c)$  that is lower than all points of  $f(x)$  that are near it.
  - A local maximum point occurs when a point,  $c$ , has an output  $f(c)$  that is greater than all points of  $f(x)$  that are near it.
  - An absolute extreme minimum occurs when a point,  $c$ , has an output  $f(c)$  that is lower than all points on  $f(x)$  for all  $x$  in the domain.
  - An absolute extreme maximum occurs when a point,  $c$ , has an output  $f(c)$  that is greater than all points on  $f(x)$  for all  $x$  in the domain.

• find absolute max/min of  $f(x) = \ln \left| \frac{x}{2+x^2} \right|$  on  $-2 \leq x \leq 2$

$$f(x) = \ln(x) - \ln(2+x^2)$$

$$f'(x) = \frac{1}{x} - \frac{1}{(2+x^2)} \cdot (2x)$$

$$= \frac{1}{x} - \frac{2x}{(2+x^2)}$$

$$\frac{2x^2 - 2x^2}{(x)(2+x^2)}$$

$$\frac{2-x^2}{(x)(2+x^2)}$$

① Simplify  
to take  $f'(x)$

$$\frac{2-x^2}{(x)(2+x^2)} \rightarrow 2-x^2=0 \\ 2=x^2 \\ (\pm\sqrt{2}=x) \quad \checkmark \text{ DOMAIN}$$

$x=0$  no roots

✓ DOMAIN

$$f(-2) \approx -1.099$$

$$f(2) \approx -1.099$$

$f(a)$

$f(b)$

$$f(0) = \lim_{x \rightarrow 0} f(x) = -\infty \quad f(c)$$

$$f(\sqrt{2}) \approx -1.04 \quad f(c)$$

$$f(-\sqrt{2}) \approx -1.04 \quad f(c)$$

② Find critical points  
(check to see if critical points are in domain)

③ Find  $f(a), f(b)$

and all  $f(c)$

NOTE:  $f(0)$

cannot be evaluated so use a limit.

④ Look for MAX/MIN and quote as points

THINK ...

which is the lowest output?  $-\infty$

BUT, is this an extreme value? NO.

SO... absolute minimum on this interval DNE.

which is the largest output?  $-1.04$

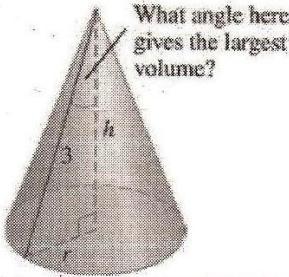
∴ There is no absolute minimum but there are 2 absolute maximums at  $\dots (\pm\sqrt{2}, -1.04)$  approx.  
(AKA maximum value is approx  $-1.04$ )

## (1a) Optimization → STEPS TO APPROACH WORD PROBLEMS WITH

1. Understand the problem. Draw a diagram Introduce notation (variables must be defined with let statements or with the use of a diagram with labels). Figure out the variable to be maximized, and use the information in the question to express this variable in terms of one other variable. *or minimized*
2. Find domain (in terms of independent variable)
3. Find the critical points
4. Find the absolute maximum or minimum of the function on its domain using one of the following:
  - 1<sup>st</sup> or 2<sup>nd</sup> derivative test
  - Closed Interval Method.
  - Using limits on open interval
5. Answer the question. Ask yourself if the answer makes sense.

## (1b) Example of an optimization question

The slant height of the cone is 3m. How large should the indicated angle be to maximize the cone's volume? Ensure to verify that the answer you give is a maximum.



- When doing step 4 as listed above:
- do 1<sup>st</sup> deriv if easy to factor (+/- chart)
  - do 2<sup>nd</sup> deriv if easy to do  $y''$
  - do closed interval method if domain is  $[a, b]$

needs to be maximized

We know: Volume of cone =  $\frac{1}{3}\pi r^2 h$

let  $\theta$  = angle



$$\text{SOH} \rightarrow \sin\theta = \frac{r}{l} \quad \text{CAH} \rightarrow \cos\theta = \frac{l}{h}$$

$$3\sin\theta = r$$

$$3\cos\theta = h$$



$$V = \frac{1}{3}\pi(3\sin\theta)^2(3\cos\theta)$$

$$V = \frac{1}{3}\pi(9\sin^2\theta)(3\cos\theta)$$

$$V = 9\pi\sin^2\theta\cos\theta \quad \text{Domain: } 0 < \theta < \frac{\pi}{2}$$

- ① Create variables (the least amount possible) and identify what needs to be maximized.

use the created variable to form an equation.

(use a trig ratio)

isolate  $r/h$  in the "angle's equations."

Then, sub into  $V =$

- ② Find domain

$$V' = 9\pi[(2)(\sin\theta)(\cos\theta)\cos\theta - \sin\theta(\sin^2\theta)] \quad \text{③ Find critical points}$$

$$= 9\pi(2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$= 9\pi\sin\theta(2\cos^2\theta - \sin^2\theta) \quad \text{PYTHAG. ID}$$

$$= 9\pi\sin\theta(2\cos^2\theta - (1-\cos^2\theta))$$

$$= 9\pi\sin\theta(3\cos^2\theta - 1)$$

$$0 = 9\pi\sin\theta(3\cos^2\theta - 1) \quad \text{(critical points: } \sin\theta = 0 = \frac{y}{r}, \cos^2\theta = \frac{1}{3}$$



$$\cos\theta = \pm \frac{1}{\sqrt{3}} = \frac{x}{r}$$

critical pt:  $\theta = 0.956$  (radians)

$$\theta_1 = 0.956 \sqrt{m}$$

$$\lim_{\theta \rightarrow 0.956} 9\pi\sin^2\theta\cos\theta = 10.883$$

$$\lim_{\theta \rightarrow 0} 9\sin^2\theta\cos\theta = 0$$

$$\lim_{\theta \rightarrow \pi/2} 9\sin^2\theta\cos\theta = 0$$

$\therefore \theta = 0.956$  is

max

- ④ use limits on an open interval to show  $0.956$

is max on domain  $0 < \theta < \frac{\pi}{2}$

$$\frac{0.956 \times 180^\circ}{\pi}$$

The angle indicated should be  $0.956$  (radians)

or approx.  $54.7^\circ$  to maximize the cone's volume.

- ⑤ Answer the question.