

Function Analysis

(2a)

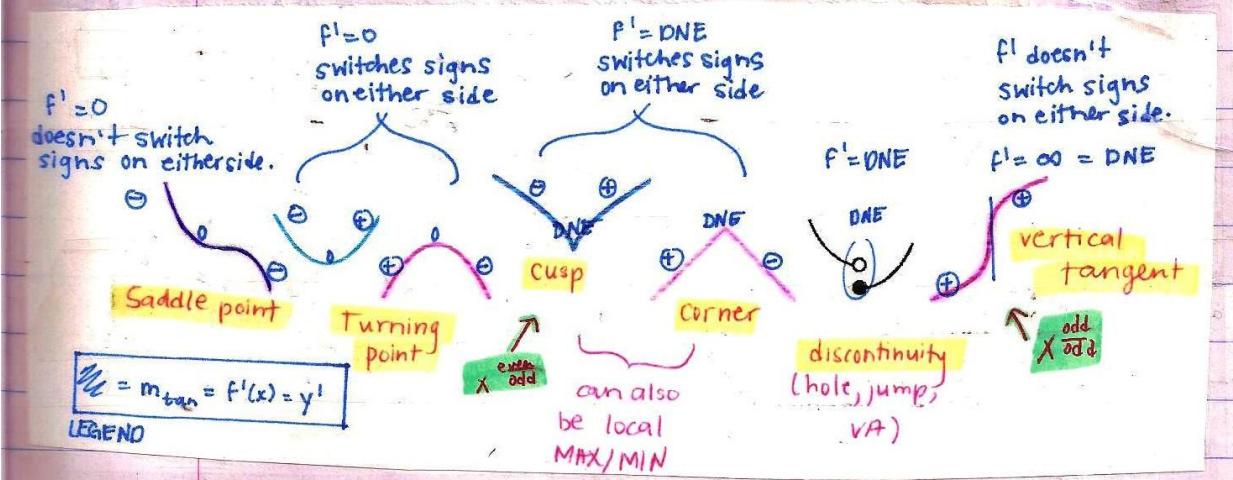
A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist. (t.p., saddle pt, cusp...)

If the graph of f lies above all of its tangents on an interval I , then it is called **concave upward** on I . If the graph of f lies below all of its tangents on I , it is called **concave downward** on I . A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward (or vice versa) at P .

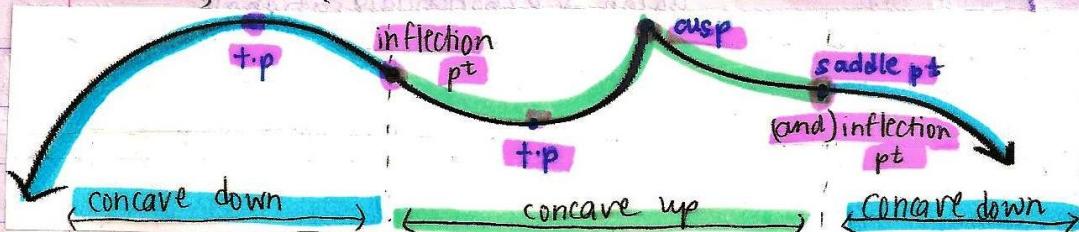
Inflection pts may be saddle pts or vertical tangents. However, for this to be true, slope must = 0 or DNE at those points.

★ look at original!

(2)b) Classifying different types of critical points



(2)c) Identifying inflection points and intervals of concavity



(2)d) Increasing/Decreasing Test:

If $f'(x) > 0$ on an interval, then f is increasing on that interval.

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Concavity Test

If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I .

If $f''(x) < 0$ for all x in I , then the graph of f is concave downward on I .

(2)e) How to check if vertex/t.p is a LOCAL max/min.

The First Derivative Test: Suppose that c is a critical number of a continuous function f .

■ f' changes from positive to negative at c , then f has a local maximum at c .

■ f' changes from negative to positive at c , then f has a local minimum at c .

■ f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c .

The Second Derivative Test: Suppose f'' is continuous near c .

■ $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .

■ $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .

Sketching

(3a) UPDATED ALGORITHM

Note that *steps can be skipped since everything else may give enough info to determine all details of the graph.

- From f
 - Factor to see if any Holes exist then state VA's and Domain (watch out for restrictions in denominators, roots and log functions)
 - Find x and y intercepts
 - Decide if there is HA or OA
 - Look at multiplicities of zeros and VA's to determine function behavior near zeros and VA's
 - *Find positive/negative intervals ie. test intervals
- From f'
 - Find critical points (classify them using 1st derivative test if you had to state intervals of inc/dec, OR using 2nd derivative test if you had to state concavity)
 - *Find increasing/decreasing intervals
- From f''
 - Find possible inflection points (decide if actually inflecting there by looking at CU/CD intervals)
 - Find concave up/down intervals
- At the end use f again to find the y values of critical and inflection points to order to sketch them.

Functions with VA's \rightarrow logs/rationals/trig

HA/OA \rightarrow exponentials/rationals

(updated)
but similar
to above

FROM f'' ① Are they inflection pts?

② Classify using 2nd derivative (critical points, that is)

*DON'T FORGET TO LOOK AT/FOR INTERCEPTS.

(3b) How to sketch and label all intercepts, asymptotes, critical points, and inflection points

• $y = (x^{1/3})(x-4)$ cannot factor any further... ① Factor to see holes/VA/domain

\therefore no holes, VA, and domain $x \in \mathbb{R}$

$$\begin{array}{l} \text{x-int} \\ 0 = x^{1/3}(x-4) \\ \downarrow \\ x=0 \end{array}$$

$$\begin{array}{l} \text{y-int} \\ y|_{x=0} = (0)^{1/3}(0-4) \\ = 0 \\ \therefore \text{pt}(0,0) \end{array}$$

HA or OA?
exponential/
rational? NO!
 \therefore no HA/OA

② Find x and y
intercepts and
classify cut/bounce
- bend if possible.

③ Decide if there
is an HA/OA.

④ Find critical
points using y'
(make = 0)
This is 1st derivative test.

$$y^1 = (x)^{1/3}(1) + (x-4)(\frac{1}{3})(x)^{-2/3}$$

$$y^1 = (x)^{-2/3} [x + \frac{1}{3}(x-4)]$$

↑ DDLCB

$$y^1 = \frac{3x + x - 4}{3(x)^{4/3}}$$

$$y^1 = \frac{4(x-1)}{3(x)^{4/3}}$$

critical pts

$$0 = \frac{4(x-1)}{3(x)^{2/3}} \rightarrow x=1, y=-3 \text{ (tip or saddle)} \quad y' = 0$$

$$-3(x)^{-2/3} \rightarrow x=0, y=0 \text{ (cusp, corner, vert. tangent)} \quad y'' = \text{DNE}$$

(use monomial version of y' to take y'')

$$y' = x^{1/3} + \frac{1}{3}(x)^{-2/3}(x-4)$$

$$y'' = \frac{1}{3}(x)^{-2/3} + \frac{1}{3}\left[(x)^{-2/3}(1) + (x-4)(-\frac{2}{3})(x)^{-5/3}\right]$$

$$y'' = \frac{1}{3}(x)^{-5/3}\left[(x) + (x) + (x-4)(-\frac{2}{3})\right]$$

DO LCD

$$y'' = \frac{6x - 2x + 8}{9(x)^{5/3}} = \frac{4(x+2)}{9(x)^{5/3}}$$

$$0 = \frac{4(x+2)}{9(x)^{5/3}} \xrightarrow{\text{possible inflection pts on } (-\infty, 0) \cup (0, \infty)}$$

$$\rightarrow x = -2 \quad \checkmark \quad y = 7.6$$

$$\rightarrow x = 0 \quad \checkmark \quad y = 0$$

⑤ Don't make a y' sign chart.
Proceed and find inflection points using y''
(make = 0)

This is 2nd derivative test.

$y'':$ $-\infty \quad -2 \quad 0 \quad \infty$

$\frac{4}{9}$	+	+	+
$x+2$	-	+	+
$x^{5/3}$	-	-	+
y''	(+)	(-)	(+)

Since at $x = -2$,
there is a CU/CD
switch, they are inflection pts.

y CU CD CU

↓ since $x=1$ is within this interval
and is not an inflection pt,
it must be a

inflection point AND

$y'(0) = \text{DNE}$, $x=0$ must be
a vertical tangent.

tip that
is also a

local min.

↳ ↑ cu = local min

⑥ NOW: Make a y''
sign chart.
(split into factors)

⑦ Use the
cu/cd intervals
to classify
critical points

⑧ Summarize
all points
found and
find their
 y -values.

(Then, sketch.
Plug into
ORIGINAL!)

POINTS

$(4, 0) \rightarrow x\text{ int}$

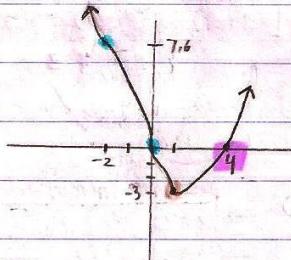
$(1, -3) \rightarrow \text{critical pt} \rightarrow \text{tip} \rightarrow \text{local min}$

$(0, 0) \rightarrow \text{inflection pt} \rightarrow \text{vertical tangent}$

$(-2, 7.6) \rightarrow \text{inflection pt}$

(use y'' chart to help
draw too!)

*TRY NOT TO MAKE
INFLECTION PTS. LOOK LIKE
SADDLE PTS. IF THEY AREN'T *



BOTH VN's
are odd multiplicity

• $y = \frac{x^3}{x^2-1} = \frac{x^3}{(x+1)(x-1)}$ ∵ no holes
VA at $x=1$ AND $x=-1$
domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

xint
(numerator)
 $x=0$ (bend)
∴ pt(0,0)
 $y|_{x=0} = \frac{0^3}{0^2-1} = \frac{0}{-1} = 0$
∴ pt(0,0)

① Factor to see
holes/VA/domain

② Find x and y
intercepts

③ Decide if there
is an HA/OA
(long division)

HA or OA? This is a rational function so YES.

$$\begin{array}{rcl} \text{degree 3} & = & 0A \\ \text{degree 2} & & \left. \begin{array}{l} x^2 + 0x - 1 \overline{) x^3} \\ \quad - x^3 - 0x^2 - x \\ \hline \quad \quad \quad x \end{array} \right\} \text{can't divide anymore} \\ \therefore 0A \text{ is } y = x & & \text{and } 0A = \text{quotient} \end{array}$$

$$y' = \frac{(x^2-1)(3x^2) - (x^3)(2x)}{(x^2-1)^2}$$

④ Find critical
points using
 y' (Make = 0)

This is 1st
derivative test.

$$0 = \frac{x^2(x^2-3)}{(x^2-1)^2} \rightarrow \begin{array}{l} \text{critical pts} \\ x=0, x=\pm\sqrt{3} \\ \rightarrow x = \pm 1 \end{array}$$

⑤ Don't make a y' sign

chart. Proceed to
finding inflection pts

using y'' . (Make = 0)

This is the 2nd derivative
test.

$$y' = \frac{x^4 - 3x^2}{(x^2-1)^2}$$

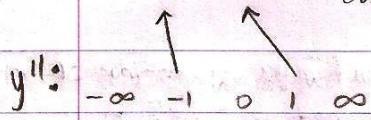
$$y'' = \frac{(x^2-1)^2(4x^3 - 6x) - (x^4 - 3x^2)(2)(x^2-1)(2x)}{(x^2-1)^4}$$

$$y'' = \frac{(x^2-1)(2x)[(x^2-1)(2x^2-3) - (x^4-3x^2)(2)]}{(x^2-1)^4}$$

$$y'' = \frac{2x[2x^4 - 3x^2 - 2x^2 + 3 - 2x^4 + 6x^2]}{(x^2-1)^3}$$

$$0 = \frac{2x(x^2+3)}{(x^2-1)^3} \rightarrow \begin{array}{l} \text{possible inflection pts} \\ x=0 \\ \rightarrow x = \pm 1 \end{array}$$

$x = \pm 1$ are critical pts
where $y' = \text{DNE}$ BUT are already our VAs.
So, we have taken them into account.



	$-\infty$	-1	0	1	∞
$2x$	-	-	+	+	
x^2+3	+	+	+	+	
$(x-1)^3$	+	=	-	+	
y''	(-)	(+)	(-)	(+)	

$x = -\sqrt{3}$ is in this interval so it must be a saddle pt. since $y' = 0$ and $y'' < 0$

local max

since $y' = 0$ and $y'' < 0$

since there is always a CD/CU switch, $x = 0, \pm 1$ are all inflection pts.

⑥ Now: Make a y'' sign chart.

⑦ Use the CU/CD intervals to classify critical pts.

FIND Y-VALUES

⑧ Summarize all points and sketch. (+asymp.)

POINTS OA at $y = x$

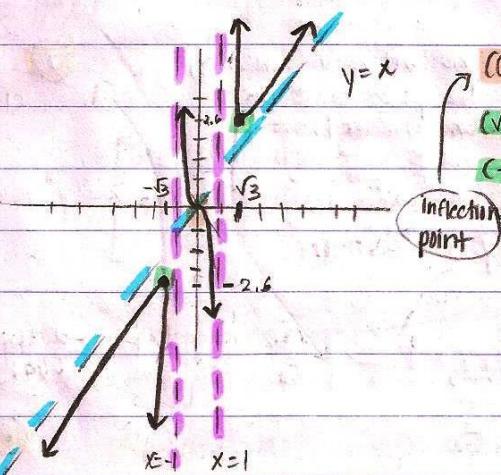
VA at $x = 1$
 $x = -1$ odd

$(0, 0) \rightarrow$ saddle pt \rightarrow x/y int

$(\sqrt{3}, 2.6) \rightarrow$ t.p. \rightarrow local min

$(-\sqrt{3}, -2.6) \rightarrow$ t.p. \rightarrow local max

* OA can be crossed in the middle!



• $y = x^2 - 8 \ln|x|$ ← too complicated to factor without technology but domain is $x \neq 0$ so: $x \in (-\infty, 0) \cup (0, \infty)$

x-int.

$0 = x^2 - 8 \ln|x|$
too complicated to solve w/o technology.

y-int

NONE
since VA at $x=0$
 $\downarrow \log x = \text{VA}$

HA or OA?

exponential or rational? NO
 \therefore no HA/OA.

② Find x/y intercepts

③ Decide if there is an HA or OA.

④ Find critical pts using $y' = 0$.
This is the 1st derivative test.

$$y' = 2x - 8 \cdot \frac{1}{x}$$

DD LCB

$$y' = 2 \frac{x^2 - 8}{x} = \frac{2(x+2)(x-2)}{x}$$

$$0 = \underline{2(x+2)(x-2)} \quad \begin{matrix} \text{critical pts} \\ \rightarrow x = -2, x = 2 \end{matrix}$$

x

$$\rightarrow x = 0 \quad \begin{matrix} \text{already} \\ \text{VA} \end{matrix}$$

$$y' = \underline{\frac{2(x^2-4)}{x}} \quad \begin{matrix} \text{VA} \\ \text{x} \end{matrix}$$

$$y'' = \frac{(x)2 \cdot (2x) - 2(x^2-4)(1)}{x^2} = \frac{4x^2 - 2x^2 + 8}{x^2} = \frac{2x^2 + 8}{x^2} = \frac{2(x^2 + 4)}{x^2}$$

⑤ Don't make a y' sign chart.
Proceed to find inflection pts.
using y'' . (make = 0)
This is the 2nd derivative test.

$$0 = \underline{\frac{2(x^2+4)}{x^2}} \quad \begin{matrix} \text{possible inflection pts} \\ \rightarrow \text{NO SOL.} \end{matrix}$$

$$\rightarrow x = 0$$

⑥ Now: Make a y'' sign chart,

y''	∞	0	∞
$x^2 + 4$	$+$	$+$	$+$
x^2	$+$	$+$	$+$
y''	\ominus	\oplus	\ominus
	CU	CU	CU

since there is
no switch in concavity,
the pt $x = 0$ is NOT
an inflection pt.
It is already a VA.

⑦ Use CU/CD intervals to
classify critical pts.

∴ always concave up on domain

$x = -2$ is a t.p and local
min since, in this interval,
it is always CU.

$x = 2$ is a t.p and local min
since, in this interval, it is always CU.

POINTS

VA at $x = 0$

$(2, -1.55)$ \nearrow t.p. and local min
 $(-2, -1.55)$ \searrow

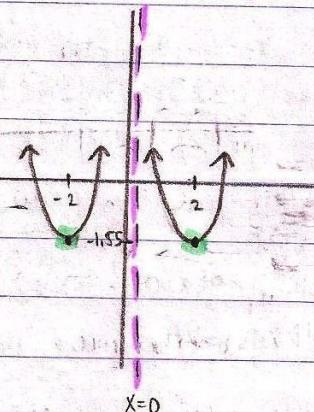
use for detail with drawing:

$y'': -\infty \rightarrow -2 \leftarrow 0 \rightarrow 2 \rightarrow \infty$

$x^2 + 4$	$+$	$+$	$+$	$+$
x^2	$+$	$-$	$-$	$+$
x	$-$	$-$	$+$	$+$
y''	\ominus	\oplus	\ominus	\oplus

y dec inc dec inc

$0 \quad \text{DNE} \quad 0$



⑧ Find y -values
of points. Then,
summarize
points and
sketch.

⑨ If needed,
complete a
sign chart
for y' to
aid in drawing.

• $y = x - 2\cos x$ on $x \in (0, 2\pi)$ ← can't factor because it's too complicated without technology
no holes/VA and domain is $x \in \mathbb{R}$

① Factor to see holes/VA/domain

x_{int} y_{int}
too complicated to factor w/o technology $y|_{x=0} = 0 - 2\cos(0) = -2(1) = -2$
 $\therefore (0, -2)$
But domain here doesn't include 0.

② Find x/y intercepts

③ Decide if HA/OA is present

HA or OA?

exponential/rational?

NO! ∵ no HA/OA

$$y^1 = 1 + 2\sin x$$

critical pts

$$(n=0) x = \frac{7\pi}{6}, x = \frac{11\pi}{6}$$

⑤ Don't make a y^1 sign chart.

Proceed to finding inflection pts using y^{11} .

(Make = 0) This is the 2nd derivative test.

$$y^1 = 1 + 2\sin x$$

$$y^{11} = 2\cos x$$

$$0 = 2\cos x$$

$$0 = \cos x$$

possible inflection pts within $(0, 2\pi)$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

⑦ Use CU/CD intervals to classify critical pts.

⑧ Find y-values and summarize points to sketch. (on next page)

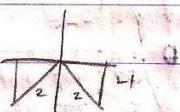
④ Find critical pts using y^1 . (Make = 0)
This is 1st derivative test.

$$y = \frac{1}{2} = \sin x$$

$$\left[x = \frac{7\pi}{6} + 2\pi n \right]$$

$$\left[x = \frac{11\pi}{6} + 2\pi n \right]$$

general sol. $\{n \in \mathbb{Z}\}$



Now, find which values of n give x-values from $(0, 2\pi)$.

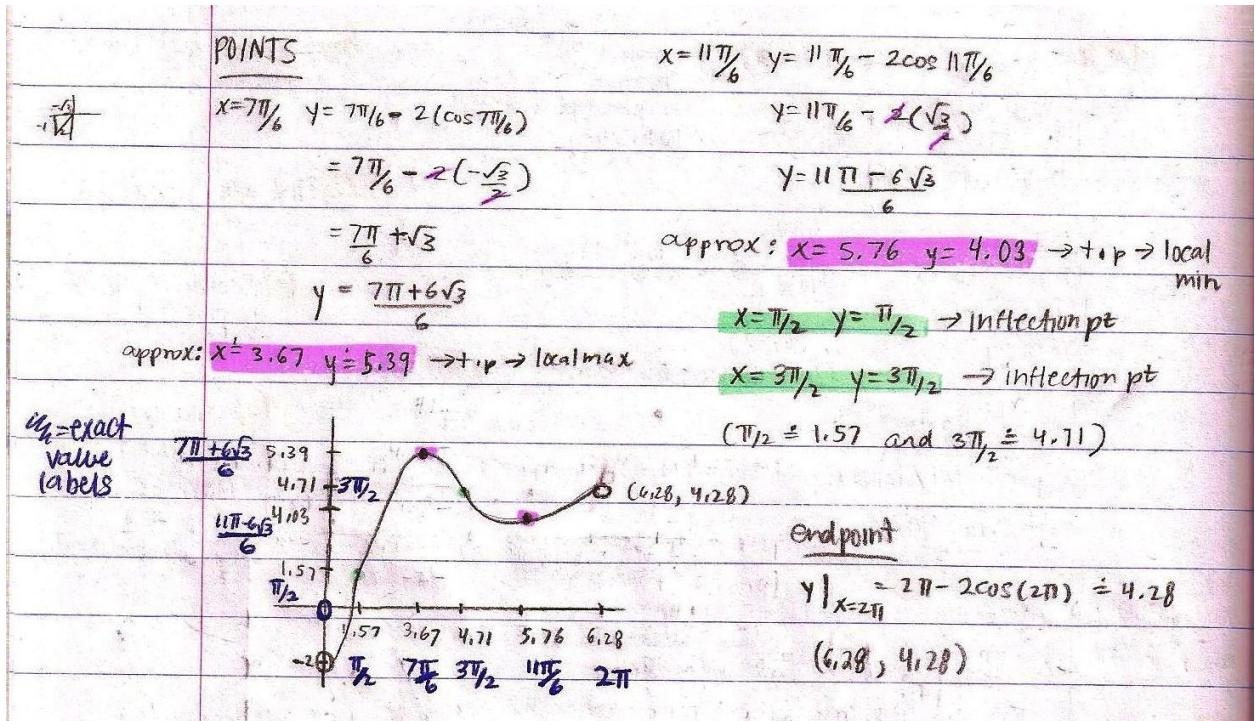
⑥ Now: Make a sign chart for y^{11} , (USING DOMAIN OF FUNCTION!).

	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π
$\cos x$	+	+	-	+
y^{11}	+	-	+	+
y	cu	cd	cu	cd

since there are switches between CU/CD, $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ are inflection pts.

$x = \frac{\pi}{2}$ is in this interval and is a tip and local min.

$x = \frac{3\pi}{2}$ is in this interval and is a tip and local max.



Characteristics of f and f'

- f is CU $\rightarrow f'$ is neg \rightarrow y -values go closer to x -axis $\rightarrow f'$ is pos \rightarrow y -values away from x -axis
- f has t.p. $\rightarrow f'$ has zero
- f has saddle (inf.pt.) $\rightarrow f'$ has zero
- f has vert.tangent (inf.pt.) $\rightarrow f'$ has VA with even symmetry
- f has cusp $\rightarrow f'$ has VA with odd symmetry
- f has inf.pt. $\rightarrow f'$ has a.t.p.
- f is CD $\rightarrow f'$ is pos \rightarrow y -values get closer to x -axis $\rightarrow f'$ is neg \rightarrow y -values go away from x -axis
- f is inc (going north) $\rightarrow f'$ is pos (above x -axis)
- f is dec (going south) $\rightarrow f'$ is neg (below x -axis)