

(7a)

## Derivatives of Exponentials and Logarithms

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a \cdot \frac{da}{dx}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

## Definition:

$e$  is the number such that  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

derivative of exp.

derivative of input

## EXPONENTIALS

the original  $x$  ln of  
the base and derivative  
of the exponent

## LOGARITHMS

1 ÷ the input and ln of the base  
THEN  $x$  derivative of the input.

(7b) Significance of the number  $e$ ...!

$e$  is a constant that makes a function and its derivative the same / "overlap".

When the base of an exponential is  $e$ , the exponential and its derivative will be the same.

$$\hookrightarrow \frac{d}{dx}(e^x) = (e^x)$$

(5a)

## Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

① Plug into definition

② Evaluate

 $\sin(x+h)$ 

using compound ID.

③ Split into 2 fractions/limits

④ Apply constant limit law

PROOF •  $\frac{d}{dx} \sin(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$

DEF

FIR

DERIVATIVE

OF  $\sin(x)$ .

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\sin x \cosh h + \cos x \sinh h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x (\cosh h - 1) + \cos x \sinh h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{(\cosh h - 1)}{h} + \cos x \lim_{h \rightarrow 0} \frac{(\sinh h)}{h} \end{aligned}$$

← COMMON FACTOR TOP

(5d) Why only memorize  $\frac{d}{dx} \sin x = \cos x$  and  $\frac{d}{dx} \cos x = -\sin x$ ?

- $\frac{d}{dx} \tan x = ?$

$$\frac{d}{dx} \frac{\sin x}{\cos x} = \cos x \left( \frac{d}{dx} \sin x \right) - \sin x \left( \frac{d}{dx} \cos x \right)$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

- $\frac{d}{dx} \csc x = ?$

$$\frac{d}{dx} \sin x^{-1} = -(\sin x^{-2})(\cos x) = -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\frac{(\cot x)(\csc x)}{\sin^2 x}$$

① use quotient ID  
then quotient rule,

② Use  $\frac{d}{dx} \sin x$

and  $\frac{d}{dx} \cos x$  to evaluate

③ pythag ID  
on top

④ reciprocal ID

① Rewrite with a negative power

② use chain rule and

$\frac{d}{dx} \sin x$  to evaluate

③ Simplify  
(separate fraction)

④ Quotient ID  
reciprocal ID

$$= \sin x(0) + \cos x(1)$$

$$= \cos x$$

$\therefore \frac{d}{dx} \sin x = \cos x$

(5) Refer to trig limits  
to evaluate the  
limits.

(6c) If the input of sine is in DEGREES, the above/all trig derivatives provided on the previous page are not true. Why?

$$\bullet \frac{d \sin x^\circ}{dx} = \frac{d \sin(\frac{\pi}{180}x)}{dx} \quad \leftarrow (x^\circ)(\frac{\pi}{180}) = \left(\frac{\pi}{180}x\right)$$

$$\frac{d}{dx} \sin\left(\frac{\pi}{180}x\right) = \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{180}(x+h)\right) - \sin\left(\frac{\pi}{180}x\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin\frac{\pi}{180}x \cos\frac{\pi}{180}h + \cos\frac{\pi}{180}x \sin\frac{\pi}{180}h - \sin\frac{\pi}{180}x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin\frac{\pi}{180}x (\cos\frac{\pi}{180}h - 1) + \cos\frac{\pi}{180}x \sin\frac{\pi}{180}h}{h}$$

$$= \sin\frac{\pi}{180}x \frac{\lim_{h \rightarrow 0} (\cos\frac{\pi}{180}h - 1)}{h} \left(\frac{\pi}{180}\right) + \cos\frac{\pi}{180}x \frac{\lim_{h \rightarrow 0} \sin\frac{\pi}{180}h}{h} \left(\frac{\pi}{180}\right)$$

multiply  
by  $\frac{\pi}{180}$   
top  
bottom to  
use trig  
limit.

$$= \sin\frac{\pi}{180}x (0) \left(\frac{\pi}{180}\right) + \cos\frac{\pi}{180}x (1) \left(\frac{\pi}{180}\right)$$

$$= \left(\frac{\pi}{180}\right) \cos\frac{\pi}{180}x$$

$\therefore$  IN DEGREES,

$$- \frac{d \sin \frac{\pi}{180}x}{dx} = \frac{\pi}{180} (\cos \frac{\pi}{180}x)$$

(6) Refer to  
trig limits  
to evaluate.

(1) Convert  
degrees to  
radians

(2) Plug into  
definition

(3) Use compound  
ID

common factor

(4) Split into  
two fractions  
and limits.

(5) Also, use  
constant limit  
law.

(6) Refer to  
trig limits  
to evaluate.

missing x

## Related Rates

Q(a) The importance of Leibniz notation for the following derivatives and questions like it.

•  $\frac{d}{da} [a^2 + 3a = b^3 - ab^2]$

① Translate what Leibniz notation is asking you to do into words.

$\frac{d}{da} [\dots]$  is saying to take the derivative of the equation with respect to "a" and treat all other variables like constants.

② Take derivative

$$\frac{d}{da}(a^2) + \frac{d}{da}(3a) = \frac{d}{da}(b^3) - \frac{d}{da}(ab^2)$$

power rule      power rule      constant rule      constant rule

$\star b = \text{constant}$

$$2a + 3 = 0 - b^2$$

$b^2 + 2a + 3 = 0$  ← this is  $\frac{d}{da}$

•  $\frac{d}{db} [a^2 + 3a = b^3 - ab^2]$  ① Translate Leibniz notation to words.

$\frac{d}{db}$  Take the derivative of the equation with respect to "b" and treat all other variables like constants.

② Take derivative

$$\frac{d}{db}(a^2) + \frac{d}{db}(3a) = \frac{d}{db}(b^3) - \frac{d}{db}(ab^2)$$

constant rule      constant rule      power rule      power rule

$\star a = \text{constant}$

$$0 + 0 = 3b^2 - a2b = 3b^2 - 2ab$$

$$0 = b(3b - 2a) ← \text{this is } \frac{d}{db}$$

$$a' = a'(x) = \frac{da}{dx}$$

This means variables  
"a" and "b" are functions

•  $\frac{d}{dx} [a^2 + 3a = b^3 - ab^2]$  if  $a(x)$  and  $b(x)$

$$\frac{d(a^2)}{dx} + \frac{d(3a)}{dx} = \frac{d(b^3)}{dx} - \frac{d(ab^2)}{dx}$$

power rule      power rule      power rule      product rule

$$2aa' + 3a' = 3b^2 2b b' - (a 2b b' + b^2 a')$$

↑      ↑      ↑      ↓

$$2aa' + 3a' = 6b^3 b' - 2abb' + b^2 a'$$

$$2aa' + 3a' - b^2 a' = 6b^3 b' - 2abb'$$

$$a'(2a + 3 - b^2) = 2bb'(3b^2 - a)$$

(1) Notice that you must do IMPLICIT differentiation here since all variables are constants.

(2) Take derivative implicitly

(3) Collect like terms, factor, simplify

\* At this point, we would sub in values for

$a', b', a, b$  if given original functions \*

### (2b) Formulas for reference

Pythagoras	$a^2 + b^2 = c^2$	Circle	$A = \pi r^2, C = 2\pi r$
Trig. Formulas	SOH, CAH, TOA	Sphere	$V = \frac{4}{3}\pi r^3, SA = 4\pi r^2$
Sine Law	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Cylinder	$V = \pi r^2 h, SA = 2\pi r^2 + 2\pi r h$
Cosine Law	$c^2 = a^2 + b^2 - 2ab \cos C$	Cone	$V = \frac{1}{3}\pi r^2 h, SA = \pi r \sqrt{r^2 + h^2}$
Area of triangle	$\frac{1}{2}bh$ or $\frac{1}{2}abs \sin \theta$	Prisms	$V = (\text{Area of base}) \times (\text{Distance between bases})$
		Similar Triangles	Ratios of corresponding sides are equal.

### (3) Related Rates → Steps to approach word problems with

1. Read the problem to determine what is given and what is required. (Pay attention to units "per" for rate of change) Introduce notation. If possible, draw a diagram.
2. Write an equation relating the variables in the problem. There can be more than one equation needed.
3. Differentiate both sides of the equation with respect to t. (Ask yourself if the quantity actually changes with time or if it's constant).
4. Substitute in the given info AFTER taking the derivative and solve for the unknown rate of change. Use another equation if there are too many unknowns still.
5. Answer the question, include units

Information in [ ] is the question and numbers here are plugged in AT THE END.

(2d) Related Rates word problem example.

- Water flows at  $8 \text{ cm}^3/\text{min}$  into a cone with a radius of 4 cm and height of 12 cm. [How fast is the water level rising when the water is 2 cm deep?]

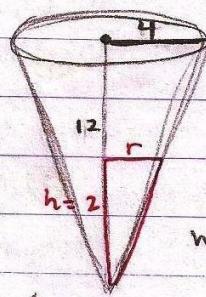
increasing  
 $\therefore \oplus$

the "water level" is represented by which variable?

Volume ( $V$ )?

Height ( $h$ )? ✓

radius ( $r$ )?



given:  $r = 4$  } of cone  
 $h = 12$

$$\frac{dv}{dt} = 8 \text{ cm}^3/\text{min}$$

want:  $\frac{dh}{dt} = ?$   $\oplus$

(1) Draw a picture and look at what is given and what is required.

tells us the "thing" is volume!

using similar triangles:  $\frac{4}{r} = \frac{12}{h}$

using volume:  $V = \frac{1}{3}\pi r^2 h$

where:  $v(t)$ ,  $h(t)$ ,  $r(t)$

$$\frac{d}{dt} \left[ V = \frac{1}{3}\pi r^2 h \right]$$

$$4h = 12r$$

$$h = 3r \quad \text{OR} \quad \frac{1}{3}h = r$$

$\frac{1}{3} \frac{dh}{dt} = \frac{dr}{dt}$  change with time.

(2) Create an equation(s) and identify which variables change with time.

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot r^2 \frac{dh}{dt} + h(2r) \frac{dr}{dt}$$

we need a value for this!! use that ↑

common factor  $\frac{dh}{dt}$

$$8 = \frac{1}{3}\pi \left[ \left(\frac{2}{3}\right)^2 \frac{dh}{dt} + (2)(2)\left(\frac{2}{3}\right) \left(\frac{1}{3} \frac{dh}{dt}\right) \right]$$

$$8 = \frac{1}{3}\pi \frac{dh}{dt} \left( \frac{4}{9} + \frac{8}{9} \right)$$

$$(3) \frac{8(3)}{\pi} = \frac{4}{3} \frac{dh}{dt} \left( \frac{3}{4} \right)$$

$$\frac{18}{\pi} = \frac{dh}{dt} \approx 5.73$$

(3) Take  $\frac{d[ ]}{dt}$

$$\frac{dv}{dt} = 8$$

$$h = 2$$

$$r = \frac{1}{3}h = \frac{2}{3}$$

$$\frac{dr}{dt} = \frac{1}{3} \frac{dh}{dt}$$

(4) Sub-in  
 $\frac{dv}{dt} = 8$   
 $h = 2$   
 $r = \frac{2}{3}$   
 $\frac{dr}{dt} = \frac{1}{3} \frac{dh}{dt}$

(5) Answer the question

$\therefore$  The water level is rising at

$5.73 \text{ cm/min}$  when the water is 2cm deep.