

Conics

① Foundation

①a) How to complete the square

• $y = -3x^2 + 2x - 5$
 $y = ax^2 + bx + c$

$$\begin{aligned} y &= -3(x^2 - \frac{2}{3}x) - 5 \\ &= -3[x^2 - 4x + (-2)^2] - (-2)^2(-3) - 5 \\ &= -3(x - 2)^2 + 7 \end{aligned}$$

① Common factor out any a -value from a and b .

② Within the brackets, add $(\frac{b}{2})^2$ and outside the brackets, subtract $(\frac{b}{2})^2$.

* Apply a -value outside brackets

③ Factor brackets as perfect square trinomial and simplify on the outside.

①b) Common Mistake with powers and roots

• $\sqrt{25 \cdot 9} = 5 \cdot 3$ BUT $\sqrt{25 - 9} \neq 5 - 3$

NOTE: $\sqrt{25 \cdot 9}$ BUT: $\sqrt{25 - 9}$

$$\begin{array}{lll} \text{exponent laws applying across } \times / \div & \begin{array}{l} = (25 \cdot 9)^{1/2} \\ = 25^{1/2} \cdot 9^{1/2} \\ = \sqrt{25} \cdot \sqrt{9} \\ = 5 \cdot 3 \\ = 15 \end{array} & \begin{array}{l} = (25 - 9)^{1/2} \\ \neq 25^{1/2} - 9^{1/2} \\ \neq \sqrt{25} - \sqrt{9} \\ \neq 5 - 3 \\ \neq 2 \end{array} \\ \text{From this point, all steps are incorrect because exponent laws don't apply across } + / - \end{array}$$

AND: $\sqrt{25 \cdot 9}$

$$= \sqrt{225}$$

$$= 15$$

HOWEVER: $\sqrt{25 - 9}$

$$= \sqrt{16}$$

$$= 4$$

①c) Solving Radical Equations

$$\sqrt{2x+5} + \sqrt{x+2} = 5$$

\swarrow EXPAND AS
 $(\sqrt{2x+5})^2 = (5 - \sqrt{x+2})^2$ \swarrow PERF. \square TRI.

$$2x+5 = 25 - 10\sqrt{x+2} + x+2$$

$$2x+5 - 25 - x-2 = -10\sqrt{x+2}$$

$$-(x-22)^2 = (10\sqrt{x+2})^2$$

$$(x-22)^2 = 100(x+2)$$

$$x^2 - 44x + 484 = 100x + 200$$

$$x^2 - 144x + 284 = 0$$

① Bring one root term over THEN square both sides for each.

② Isolate remaining root and square both sides again

③ Expand + Simplify

$$x^2 - 144x + 284$$

$$x = \frac{144 \pm \sqrt{19600}}{2}$$

$$x = 142 \quad x = 2$$

(4) Factor (quad form.)

(5) WE ARE NOT
DONE!

When we square both sides, we are changing the domain restrictions on the original. So, test both answers.

when $x = 142$

$$\sqrt{2(142)+5} + \sqrt{142+2} = 5$$

$$17 + 12 = 5$$

$$29 \neq 5$$

so $x = 142$ is NOT a valid solution.

when $x = 2$

$$\sqrt{2(2)+5} + \sqrt{2+2} = 5$$

$$3 + 2 = 5$$

$$5 = 5$$

so $x = 2$ is a (only) valid solution.

Shown above is how to deal with extraneous solutions.

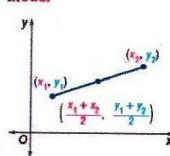
(1d)

Key Concept

Words If a line segment has endpoints at (x_1, y_1) and (x_2, y_2) , then the midpoint of the segment has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Symbols midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

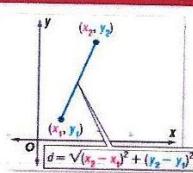
Midpoint Formula



Key Concept

Words The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distance Formula



...so also each conic has a "typical" equation form, sometimes along the lines of the following:

parabola: $Ax^2 + Dx + Ey = 0$

circle: $x^2 + y^2 + Dx + Ey + F = 0$

ellipse: $Ax^2 + Cy^2 + Dx + Ey + F = 0$

hyperbola: $Ax^2 - Cy^2 - Dx - Ey + F = 0$

These equations can be rearranged in various ways, and each conic has its own special form that you'll need to learn to recognize, but some characteristics of the equations above remain unchanged for each type of conic. If you keep these consistent characteristics in mind, then you can run through a quick check-list to determine what sort of conic is represented by a given quadratic equation.

General Form

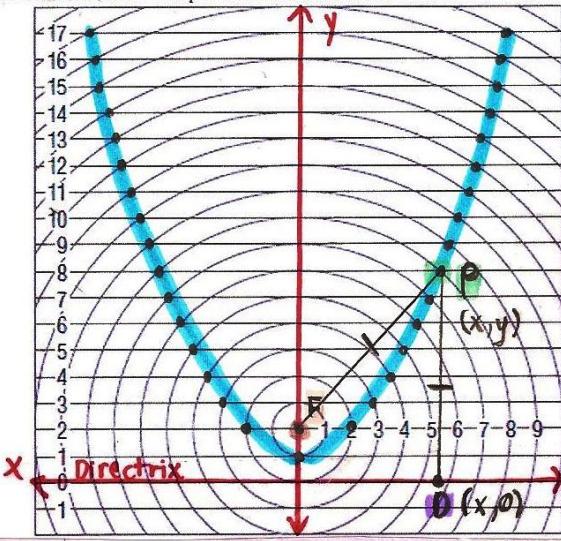


Parabolas

(2a) A parabola can also be defined as the set of all points in a plane that are the same distance from the focus and directrix.

(2b)

Mark the point at the intersection of circle 1 and line 1. Mark both points that are on line 2 and circle 2. Continue this process, marking both points on line 3 and circle 3, and so on. Then connect the points with a smooth curve.



On the diagram place y-axis at the centre of inner circle and x-axis at line 0. Mark off the centre of the inner circle as the focus with $F(0, 2)$, any general point on the parabola as $P(x, y)$ and right underneath the point P label the point on the directrix with $D(x, 0)$. Using the relationship $PF=PD$ and the distance formula show how

$$PF = \sqrt{(x-0)^2 + (y-2)^2} \text{ and } PD = \sqrt{(x-x)^2 + (y-0)^2}$$

$$\text{becomes } y = \frac{1}{4}x^2 + 1$$

$$D = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

$$PF = PD$$

$$\sqrt{(x-0)^2 + (y-2)^2} = \sqrt{(x-x)^2 + (y-0)^2}$$

$$(\sqrt{(x)^2 + (y-2)^2})^2 = (\sqrt{(y)^2})^2$$

$$x^2 + y^2 - 4y + 4 = y^2$$

$$x^2 + y^2 - 4y + 4 - y^2 = 0$$

$$x^2 - 4y + 4 = 0$$

$$\frac{x^2 + 4}{4} = y$$

$$\frac{x^2 + 4}{4} = y$$

$$\boxed{\frac{1}{4}x^2 + 1 = y}$$

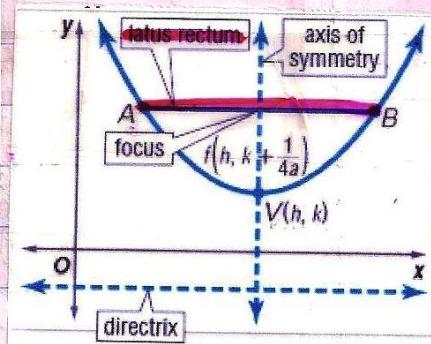
← Based on definition, equation of a parabola can be found like this.

(2c)

Concept Summary

Information About Parabolas

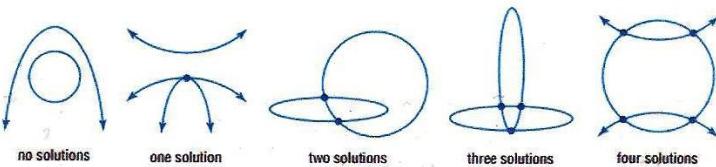
Form of Equation	$y = a(x-h)^2 + k$	$x = a(y-k)^2 + h$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$(h, k + \frac{1}{4a})$	$(h + \frac{1}{4a}, k)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Direction of Opening	upward if $a > 0$, downward if $a < 0$	right if $a > 0$, left if $a < 0$
Length of Latus Rectum	$\frac{1}{a}$ units	$\frac{1}{ a }$ units



Solving Systems and Circles

(3a) Why solving a system of equations where x/y has a square may yield more, one, two, three, or four solutions.

If the graphs of a system of equations are two conic sections, the system may have zero, one, two, three, or four solutions. Some of the possible situations are shown below.



(3b) Ex. of solving a system

- $x^2 - 4y^2 = 9 \quad \#(1)$ hyperbola

$$4y - x = 3 \quad \#(2) \quad \text{line}$$

$$4y - 3 = x$$

$$\#(1) \quad x^2 - 4y^2 = 9$$

$$(4y - 3)^2 - 4y^2 = 9$$

$$16y^2 - 24y + 9 - 4y^2 = 9$$

$$12y^2 - 24y = 0$$

$$12y(y - 2) = 0$$

$$y=0 \quad y=2$$

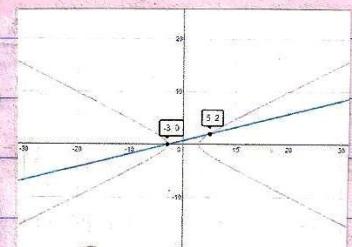


$$4(0) - 3 = x \quad 4(2) - 3 = x$$

$$-3 = x$$

$$8 - 3 = x$$

$$5 = x$$



(System is of a line and hyperbola)

① Isolate x (easier) in equation
#(2)

② Substitute into equation
#(1)

③ Expand, cLT, factor
for y -values

④ Plug into equation

#(1) or #(2) to
solve for corresponding
 x -value,

∴ the solutions to this system are

$(-3, 0)$ and $(5, 2)$

⑤ State solutions to

System in \therefore
statement

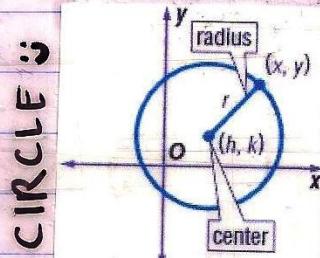
(3c) The definition of a circle is a set of all points in a plane that are equidistant from a given point in the plane called the center. Any segment whose endpoints are the center and a point on the circle is the radius of the circle.

(3d)

Key Concept

Equation of a Circle

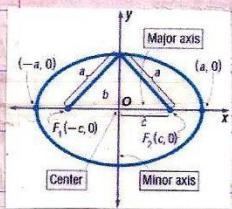
The equation of a circle with center (h, k) and radius r units is $(x - h)^2 + (y - k)^2 = r^2$.



Ellipses

(4a) The definition of an ellipse is the set of all points in a plane such that the sum of the distances from two fixed points is constant. The two fixed points are called the foci of the ellipse.

ELLIPSE



(4b)

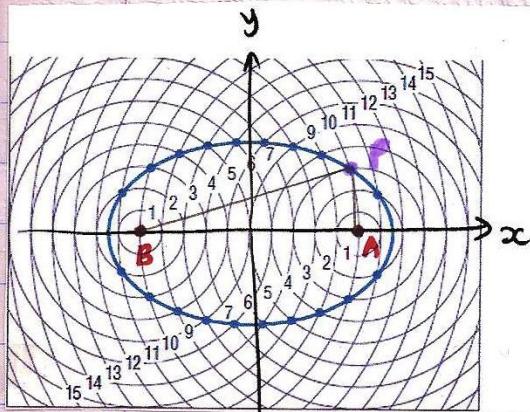
Key Concept Equations of Ellipses with Centers at the Origin

Standard Form of Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

Key Concept Equations of Ellipses with Centers at (h, k)

Standard Form of Equation	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$

(4c)



Check how each of the intersections between circle 11 and circle 2, then circle 10 and circle 3, etc. are marked off to create the ellipse.

(Each time the sum between circle numbers should be constant at 13)

Now mark off the foci with **A** (5, 0) and **B** (-5, 0), and the general point on the ellipse as **P** (x, y).

Using the relationship $PA + PB = 13$ and the distance formulas develop the following equation

$$\text{of the ellipse: } \frac{4}{169}x^2 + \frac{4}{69}y^2 = 1$$

$$P(x, y) \quad A(5, 0) \quad B(-5, 0)$$

$|PA| + |PB| = 13$ using $D = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

$$\sqrt{(x-5)^2 + (y-0)^2} + \sqrt{(x+5)^2 + (y-0)^2} = 13$$

$$(\sqrt{(x-5)^2 + (y)^2})^2 = (13 - \sqrt{(x+5)^2 + (y)^2})^2$$

$$(x-5)^2 + y^2 = 169 - 26\sqrt{(x+5)^2 + y^2} + (x+5)^2 + y^2$$

$$(x-5)^2 - (x+5)^2 + y^2 - y^2 - 169 = -26\sqrt{(x+5)^2 + y^2}$$

$$(20x-169)^2 - 26\sqrt{(x+5)^2 + y^2} = 0$$

$$400x^2 + 6760x + 28561 = 676(x^2 + 10x + 25 + y^2)$$

$$400x^2 + 6760x - 676x^2 - 6760x - 676y^2 = 16900 - 28561$$

* move terms to other side to try and get = 1

$$-276x^2 - 676y^2 = -11661$$

$$-(276x^2 + 676y^2) = -11661$$

$$\frac{276x^2 + 676y^2}{11661} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{4}{169}x^2 + \frac{4}{69}y^2 = 1}$$

① Sub-in values to distance formula then simplify and make RS = 1 to find the equation of the ellipse.

Rough:

$$(x-5)^2 - (x+5)^2$$

$$400x^2 + 6760x + 28561 - 676x^2 - 6760x - 676y^2 = 16900 - 28561$$

$$= -20x$$

NOTE:

$$a^2 \neq \frac{4}{169} \quad b^2 \neq \frac{4}{69}$$

This is the standard form equation of the ellipse that comes from definition and using the distance formula.

(1d) Verify all ellipses satisfy: $a^2 > b^2$ and $c^2 = a^2 - b^2$

(using example above) FIRST? identify a, b .

$$a^2 > b^2$$

$$c^2 = a^2 - b^2$$

$$\frac{169}{4} > \frac{69}{4}$$

$$c^2 = \frac{169}{4} - \frac{69}{4}$$

$$42.25 > 17.25$$

$$c^2 = \frac{100}{4} = 25$$

✓

$$\sqrt{c^2} = \sqrt{25}$$

✓

$$c = \pm 5$$

Is this true? (looking at ellipse's foci...)

YES!

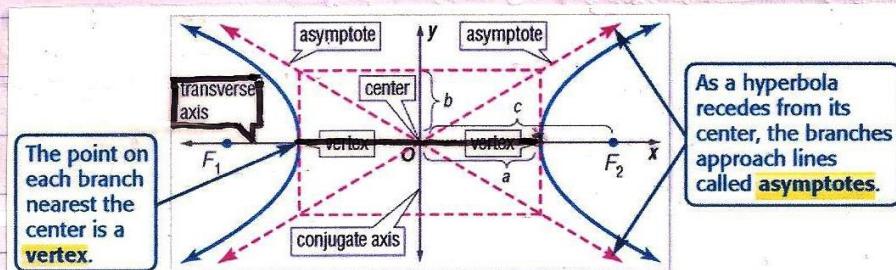
"c" is defined as the distance from the centre to each focus. The foci of this ellipse

are at $x=5$ and $x=-5$ $\therefore c^2 = a^2 - b^2$ is true.

Hyperbolas

(5a) The definition of a hyperbola is the set of all points in a plane such that the absolute value of the differences from two points (called the foci) is constant.

(5b)



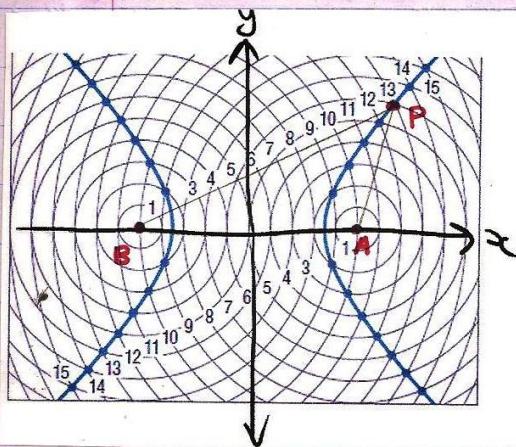
Key Concept Equations of Hyperbolas with Centers at the Origin

Standard Form of Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Vertices	$(a, 0), (-a, 0)$	$(0, a), (0, -a)$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units
Equations of Asymptotes	$y = \pm \frac{b}{a}x$	$y = \pm \frac{a}{b}x$

Key Concept Equations of Hyperbolas with Centers at (h, k)

Standard Form of Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Equations of Asymptotes	$y - k = \pm \frac{b}{a}(x - h)$	$y - k = \pm \frac{a}{b}(x - h)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Foci	$(h \pm c, k)$	$(h, k \pm c)$

(5c)



Check how each of the intersections between circle 9 and circle 2, then circle 10 and circle 3, etc. are marked off to create the hyperbola.

(Each time the difference between circle numbers should be constant at 7)

Now mark off the foci with $A(5, 0)$ and $B(-5, 0)$, and the general point on the ellipse as $P(x, y)$. Using the relationship $|PA - PB| = 7$ and the distance formulas (start with $|PA - PB| = 7$ or $-PA + PB = 7$) and develop the following equation of the ellipse:

$$\frac{4}{49}x^2 - \frac{4}{51}y^2 = 1$$

Using the distance formula and relationships for hyperbolas, we are able to come up with an equation for the hyperbola.

P(x, y)

A(5, 0)

B(-5, 0)

$$|PA - PB| = 7$$

$$PA - PB = 7$$

use $\textcircled{1}/\textcircled{2}$ version

① Sub-in pts to
distance formula

$$\sqrt{(x-5)^2 + (y-0)^2} - \sqrt{(x+5)^2 + (y-0)^2} = 7$$

$$\left(\sqrt{(x-5)^2 + y^2} \right)^2 = \left(7 + \sqrt{(x+5)^2 + y^2} \right)^2$$

$$x^2 - 10x + 25 + y^2 = 49 + 14\sqrt{(x+5)^2 + y^2} + x^2 + 10x + 25 + y^2$$

$$-10x + 25 + y^2 - 49 - x^2 - 10x - 25 - y^2 = 14\sqrt{(x+5)^2 + y^2}$$

$$-20x - 49 = (4\sqrt{(x+5)^2 + y^2})^2$$

$$400x^2 + 1960x + 2401 = 196(x^2 + 10x + 25 + y^2)$$

$$400x^2 + 1960x + 196x^2 + 1960x + 196y^2 = 4900 - 2401$$

$$\frac{204x^2 - 196y^2}{2499} = \frac{2499}{2499}$$

$$\boxed{\frac{4}{49}x^2 - \frac{4}{51}y^2 = 1}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(5d) Verify all hyperbolas satisfy: $c^2 = a^2 + b^2$

(using example above)

FIRST: identify a, b

NOTE:
 $a^2 = \frac{4}{49}$
 $b^2 = \frac{4}{51}$

$$c^2 = a^2 + b^2$$

$$c^2 = \frac{4}{4} + \frac{51}{4}$$

$$c^2 = \frac{100}{4}$$

$$c^2 = 25$$

$$\begin{cases} \frac{x^2}{(\frac{4}{49})} & \therefore a^2 = \frac{4}{4} \\ \frac{y^2}{(\frac{51}{4})} & \therefore b^2 = \frac{51}{4} \end{cases} \quad a = \frac{2}{7}, \quad b = \frac{\sqrt{51}}{2}$$

$c = \pm 5$ IS THIS TRUE? Looking at the foci of the hyperbola,
YES!

"c" is defined as being the distance from the centre to each focus.
The foci of the ellipse are at $x = \pm 5 \therefore c = \pm 5$ is true and
 $c^2 = a^2 + b^2$ is TRUE.