

5b) "SOME" SUM PROPERTIES aka: Summation formulas

•  $\sum_{i=1}^n k a_i = k \sum_{i=1}^n a_i$

•  $\sum_{i=1}^n c = \underbrace{c + c + c + \dots + c}_{n \text{ # of times}} = cn$

•  $\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$

•  $\sum_{i=1}^n ar^{i-1} = \frac{a(r^n - 1)}{r - 1}$

arith. sum formula  $\rightarrow \sum_{i=1}^n a + (i-1)d = \frac{n[2a + (i-1)d]}{2}$

geo. sum formula

•  $\sum_{i=1}^n i = \frac{n(n+1)}{2} = \frac{n^2 + n}{2}$

•  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

\* special case of arithmetic series

$= \frac{2n^3 + 3n^2 + n}{6}$

•  $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4 + 2n^3 + n^2}{4}$

NOTES:

$\rightarrow$  SIGMA distributes across addition and subtraction.

$\sum_{n=1}^{60} n + 2 \Rightarrow \sum_{n=1}^{60} n + \sum_{n=1}^{60} 2$

$\rightarrow$  When you have multiplication, you can pull it out as a coefficient.

$\sum_{n=1}^{60} 2n \Rightarrow 2 \sum_{n=1}^{60} n$

$\rightarrow$  What you are essentially trying to do in order to use summation formulas is isolate the variable and then look at what summation formula best resembles it.

⑤ Example of how to evaluate a sum using sum properties.

$$\sum_{i=1}^{60} i^3 - 4i + 3 - 2^i$$

① Distribute the sigma to every term.

$$\sum_{i=1}^{60} i^3 - \sum_{i=1}^{60} 4i + \sum_{i=1}^{60} 3 - \sum_{i=1}^{60} 2^i$$

② Simplify by using the

$$\sum_{i=1}^{60} i^3 - 4 \sum_{i=1}^{60} i + \sum_{i=1}^{60} 3 - \sum_{i=1}^{60} 2^i$$

$\sum_{i=1}^n ka_i$  property.

(pull out any #s being multiplied as a coefficient)

$$\sum_{i=1}^{60} i^3$$

$$i = 60 - 1 + 1 \\ i = 60$$

$$4 \sum_{i=1}^{60} i$$

$$\sum_{i=1}^{60} 3$$

③ For each term, determine what sum property to use to simplify.

$$\frac{n^4 + 2n^3 + n^2}{4}$$

$$\frac{n^2 + n}{2}$$

$$\frac{cn}{3(60)}$$

④ Solve separately then put together and CLT.

$$\frac{60^4 + 2(60)^3 + 60^2}{4}$$

$$\frac{60^2 + 60}{2}$$

$$= 180$$

$$= 3348900$$

\* DON'T FORGET TO MULTIPLY BY 4

$$\sum_{i=1}^{60} 2^i$$

\* use geometric formula

$$4(1830)$$

$$\frac{a(r^n - 1)}{r - 1}$$

$$= 7320$$

$$a = 2 \quad r = 2 \quad \frac{2(2^{60} - 1)}{2 - 1}$$

$$\approx 2.3 \times 10^{18}$$

PUT TOGETHER AND EVALUATE:

$$3348900 - 7320 + 180 - 2.3 \times 10^{18} \approx -2.3 \times 10^{18}$$

## 5e) Descartes's Rule of Signs

### Descartes's Rule of Signs:

The number of positive (possible) roots of  $f(x)$  is equal to the number of sign changes in the list of coefficients. More possibilities can be found by skip counting down by 2.

The number of negative (possible) roots of  $f(x)$  is equal to the number of sign changes in the list of coefficients in  $f(-x)$ . More possibilities can be found by skip counting down by 2.

When do we use this?

Descartes's rule can be used with the rational root theorem.

The rational root theorem gives you a list of  $\pm$  possible roots.

If you can find out how many possible  $\pm$  roots there are using Descartes's rule, you can try specific numbers first.

- using the example from 5c, as shown; using the rational root theorem was repetitive and long when trying to find the two other roots. Here is how Descartes's rule could have been used.

$$f(x) = 2x^3 - 9x^2 + x + 12$$

sign changes in  $f(x) = 2$

$$\therefore \# \text{ of possible positive zeros} = 2, 0$$

① Find # of possible positive roots

$$f(-x) = 2(-x)^3 - 9(-x)^2 + (-x) + 12$$

$$f(-x) = -2x^3 - 9x^2 - x + 12$$

sign changes in  $f(-x) = 1$

$$\therefore \# \text{ of possible negative zeros} = 1$$

② Find # of possible negative roots

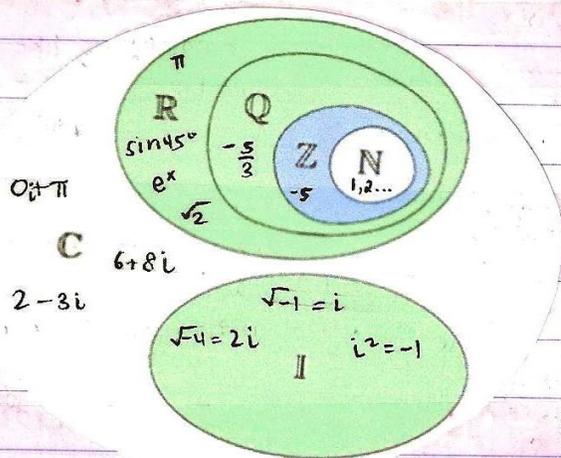
Knowing this about the zeros would encourage us to try the positive versions of possible roots in our list first.

(or, just try more positives than negatives)

Imaginary roots/zeros of a polynomial ARE NOT x-intercepts!!!!

Imaginary #s and complex roots of polynomials

8a REALMS OF NUMBERS



Types of #s

N = Natural Numbers

Z = Integers

Q = Rational Numbers

R = Real Numbers

I = Imaginary Numbers

ALL types of #s are/ can be considered

C = Complex Numbers

8b) Imaginary number → definitions

$i = \sqrt{-1}$      $i^2 = -1$

$i$  is an imaginary number used to factor/represent polynomials that have no real roots.

NOTE: Conjugate of a single imaginary # is just 0 - bi

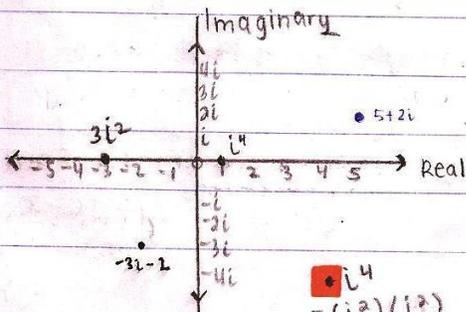
Complex number → definition + conjugate

$z = a + bi$  This is a complex number. A complex number is made up of two parts. A real part and an imaginary part.  $a$  = real  $bi$  = imaginary

$\bar{z}$  or  $z^*$  This represents the conjugate of a complex number.  $\bar{z}$  of the standard complex number above is  $\bar{z} = a - bi$

How to sketch complex numbers on the Argand (complex) plane

On this plane, your x-axis is real #s but your y-axis is imaginary #s ( $-i, 2i, -3i, \dots, i, 2i, 3i$ )



$5+2i$  To sketch, move 5 units on the real axis, and 2i units on the imaginary axis.

$-3i-2$  To sketch, move  $-3i$  units on imaginary axis and move  $-2$  units on the real axis.

$i^4 = (i^2)(i^2) = (-1)(-1) = 1$  on real axis. To sketch, move 1 unit

$3i^2 = 3(-1) = -3$  on real axis. To sketch, move 3 units

## 80) Operations with complex numbers

**Adding/Subtracting complex numbers** This process uses the concept of collecting like terms.

•  $(3+4i) + (7-5i)$

$$3+4i + 7-5i$$

$$3+7 + 4i-5i$$

$$= 10 - i$$

① Get rid of the brackets

② Collect like terms + simplify

•  $(-2+7i) - (-4+7i)$

$$-2+7i + 4 - 7i$$

$$-2+4 + 7i-7i$$

$$= 2$$

① Get rid of the brackets AND distribute subtraction!

② Collect like terms + simplify

## Multiplying/Dividing complex numbers

Process of multiplication: FOIL as always, or expand normally (ie Pascal's  $\Delta$ ). Then, sub-in values for  $i^2$  or  $i$  to help cancel at end.

NOTE: unlike multiplying radicands to simplify radicals, you cannot do this. First, change the radical term into an imaginary number THEN simplify.

•  $\sqrt{-36} \times \sqrt{-9}$

$$\sqrt{-36} \times \sqrt{-9} \neq \sqrt{324}$$

$$\neq 18 \quad \times$$

$$\sqrt{-36} \times \sqrt{-9} = \sqrt{-1} \sqrt{36} \times \sqrt{-1} \sqrt{9}$$

$$= 6i \times 3i$$

$$= 18i^2$$

$$= 18(-1)$$

$$= -18 \quad \checkmark$$

•  $(1+i\sqrt{3})^3$

$$\begin{matrix} 1 & 2 & 1 \\ 3 & 3 & 1 \end{matrix}$$

① Use Pascal's  $\Delta$  to expand

ROUGH:

$$(\sqrt{3})^3 = (3^{\frac{1}{2}})^3$$

$$= 3^{\frac{3}{2}}$$

$$= \sqrt{3^3}$$

$$= \sqrt{27}$$

$$= \sqrt{3 \cdot 3 \cdot 3}$$

$$= 3\sqrt{3}$$

$$1(1)^3(i\sqrt{3})^0 + 3(1)^2(i\sqrt{3})^1 + 3(1)^1(i\sqrt{3})^2 + 1(1)^0(i\sqrt{3})^3$$

$$= 1 + 3i\sqrt{3} + 3i^2(3) + i^3(3\sqrt{3})$$

$$= 1 + 3i\sqrt{3} + 9(-1) + 3i^3\sqrt{3}$$

② Now, use definitions

$$= 1 + 3i\sqrt{3} + 9(-1) + 3(i^2)(i)\sqrt{3}$$

$$= 1 + 3i\sqrt{3} - 9 - 3i\sqrt{3}$$

③ Notice like terms cancel

so sometimes, leave  $i$  since it is  $\sqrt{-1}$  and may end up cancelling.

$$= 1 - 9$$

$$= -8$$

NOTE: When you multiply complex numbers that are conjugates, YOU WILL ALWAYS GET A REAL NUMBER AS YOUR PRODUCT.

$(7+2i)(7-2i)$       ① FOIL (Middle terms cancel because this is a difference of squares)  
 $49 - 14i + 14i - 4i^2$   
 $= 49 - 4i^2$   
 $= 49 - 4(-1)$       ② Sub-in definition of  $i^2$   
 $= 49 + 4$   
 $= 53$  ← REAL NUMBER

Process of division: The problem must be written in fraction form to recognize the conjugate of the denominator. The conjugate of the denominator must be used to remove  $i$  from the denominator and allow division process to continue as normal. Using knowledge of conjugates being multiplied, we know that after multiplying conjugates, the denominator will become a real

$\frac{\sqrt{2}}{\sqrt{2}+3i} \cdot \frac{(\sqrt{2}-3i)}{(\sqrt{2}-3i)}$       ① Identify conjugate and multiply numerator and denominator by conjugate.  
 ← DIFF OF SQ  
 $= \frac{2 - 3i\sqrt{2}}{2 - 9i^2}$       ② Use definition of  $i^2$  to continue simplifying  
 $= \frac{2 - 3i\sqrt{2}}{2 - 9(-1)} = \frac{2 - 3i\sqrt{2}}{2 + 9} = \frac{2 - 3i\sqrt{2}}{11}$   
 ↑ REAL NUMBER

NOTE:  $\left(\frac{5}{2i}\right)$  For this example, the conjugate would be  $0 - 2i$ , which would simplify to  $-2i$ . However, doing this will force you to have to re-simplify later. So, for monomial complex numbers, use  $\pm i$  as conjugate.

$\frac{5}{2i} \rightarrow \frac{5}{2i} \cdot \frac{-2i}{-2i} = \frac{-10i}{-4i^2} = \frac{-10i}{-4(-1)} = \frac{-10i}{4} = \frac{-5i}{2}$   
 $\downarrow$   
 $\frac{5}{2i} \cdot \frac{-i}{-i} = \frac{-5i}{-2i^2} = \frac{5i}{2(-1)} = \frac{-5i}{2}$       ∴ Use  $\pm i$  for monomial divisors, it's easier. 😊

WHEN DIVIDING: proper form = NO  $i$  in the denominator!

**Powers of  $i$**  (General step) → divide your term into  $i^2$  terms as many times as possible.

$i^5 = (i^2)(i^2)(i) = (-1)(-1)(i) = 1$        $i^{24} = (i^2)^{12} = (-1)^{12} = 1$        $i^{35} = (i^{34})(i) = (i^2)^{17}(i) = (1)^{17}(i) = i$        $-i(2i)^2 = (-i)(8)(i^2) = -8(i^2)(i^2) = -8(-1)(-1) = -8$   
 $= 1$        $= 1$        $= -i$        $= -8$

- keep exact values when doing this and rationalize!
- remember to  $\sqrt{\quad}$  the coefficients of the real/imaginary parts.

### Finding the Absolute Value of complex numbers

general formula:  $|z| = \sqrt{a^2 + b^2}$

↑  
aka  $|a+bi|$

Think of the absolute value of a complex # as its distance from the origin on the complex plane.

•  $|\sqrt{17} - i\sqrt{18}|$

$\sqrt{17} - i\sqrt{18} = z$

$|z| = \sqrt{a^2 + b^2}$

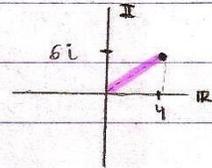
$a = \sqrt{17}$     $b = -\sqrt{18}$

↑  
It doesn't matter since it's being squared.

① Write down formula + define your complex # as  $z$ .

② Locate  $a, b$

To find this distance, use the pythag. theory and your real # values for  $a, b$ .



$|z| = \sqrt{(\sqrt{17})^2 + (-\sqrt{18})^2}$

$= \sqrt{17+18}$

$= \sqrt{35}$

③ Plug into formula and solve.

$\therefore$  absolute value of  $\sqrt{17} - i\sqrt{18} = \sqrt{35}$

How to factor a polynomial that has non-real roots

### ④ Theorems

**FUNDamental Theorem of Algebra:** A degree  $n$  polynomial will have  $n$  complex roots.

This theorem can be applied when polynomials are being factored. Sometimes, when they aren't factorable using real numbers, imaginary numbers are used.  $\therefore$  a polynomial ALWAYS has  $n$  roots (but sometimes, the  $n$  # of roots will include imaginary roots!)

•  $f(x) = 2x^3 - x^2 + 2x - 1$

Rational Root Th:  $\pm p/q$  where  $p \in \pm 1$ ,  $q \in \pm 1, \pm 2$

Factor Th:  $f(\frac{1}{2}) = 0 \therefore (x - \frac{1}{2})$  is a factor

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 2 & -1 \\ & \downarrow & & & \\ \hline & 2 & 0 & 2 & 0 \end{array}$$

$= 2x^2 + 2$

$= 2(x^2 + 1)$

① Use the rational root theorem and factor theorem to find at least one factor.

② Divide your polynomial  $f(x)$  to find your quotient that needs to be factored.

③ Recognize that this cannot be factored any further. (sum of squares). But, since this polynomial MUST have 3 roots, we must factor over imaginary numbers.

according to FUN theorem of algebra! →

b-value placeholder  
not necessary, but  
helps! :)

$$x^2 + 0x + 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{0 \pm \sqrt{0 - 4(1)(1)}}{2}$$

$$x = \frac{0 \pm \sqrt{-4}}{2}$$

$$x = \frac{0 \pm \sqrt{4}i}{2} = \frac{0 \pm 2i}{2} = \frac{0}{2} \pm \frac{2i}{2} = \boxed{0 \pm i}$$

(Distribute  
denom to  
get  $a + bi$ )

(4) To factor the sum of squares over imaginary #s, use the quadratic formula on  $(x^2 + 1)$

(5) This is when imaginary numbers must be used since  $\sqrt{-4}$  is otherwise impossible.

↑  
THE TWO IMAGINARY ROOTS  
NEEDED TO SATISFY  
THE FUN THEOREM OF  
ALGEBRA!

$$f(x) = (x + \frac{1}{2})(2)(x - (0 + i))(x - (0 - i))$$

$$f(x) = (2x - 1)(x - i)(x + i)$$

(6) Write your answer in factored form

WHY DOES IT  
SATISFY  
FUN THEOREM  
OF ALGEBRA?

original degree = 3

does this polynomial have  $n$  complex roots? YES ✓

**Irrational Root Theorem:** If  $a + \sqrt{b}$  is a root of a polynomial with rational coefficients, then  $a - \sqrt{b}$  must also be a root.

**Imaginary Root Theorem:** If  $a + bi$  is a root of a polynomial with rational coefficients, then  $a - bi$  must also be a root.

Finding the equation of a polynomial given some roots

- Find the equation of lowest degree with integral coefficients that has zeros at  $-2i$  and  $2 + 2\sqrt{2}$ . (Ignore  $a$ -value)

Irrational  
Root Th.

if  $2 + 2\sqrt{2}$  is a root, then  $2 - 2\sqrt{2}$  must also be a root.

(1) Apply the irrational root theorem and imaginary root theorem because the two roots we are given are imaginary and irrational

Imaginary  
Root Th.

if  $-2i$  is a root, then  $2i$  must also be a root.

$$y = (x - (2 + 2\sqrt{2}))(x - (2 - 2\sqrt{2}))(x - 2i)(x + 2i)$$

$$y = (x - 2 - 2\sqrt{2})(x - 2 + 2\sqrt{2})(x - 2i)(x + 2i)$$

At this point, you have found a factored form equation. However, to represent this using ONLY integral coefficients, continue factoring this.

② Notice that there are 4 roots, therefore the equation of the lowest degree will be quartic, 4th degree.

Now, write roots in factored form. Then, CLT within brackets.

$$y = (x - 2 - 2\sqrt{2})(x - 2 + 2\sqrt{2})(x - 2i)(x + 2i)$$

$$\underbrace{\quad}_{x - y} \quad \underbrace{\quad}_{x + y} \quad \underbrace{\quad}_{x - y} \quad \underbrace{\quad}_{x + y}$$

③ Factor as a difference of squares. (in other words, reverse factor)

$$y = [(x - 2)^2 - (2\sqrt{2})^2][(x^2 - (2i)^2)]$$

$$x^2 - y^2 \quad x^2 - y^2$$

④ CLT within brackets after expanding.

$$y = (x^2 - 4x + 4 - 8)(x^2 - 4(-1))$$

⑤ Sub-in for  $i^2$  to simplify further

$$y = (x^2 - 4x - 4)(x^2 + 4)$$

At this point, you have successfully factored while representing the polynomial with only integral coefficients.

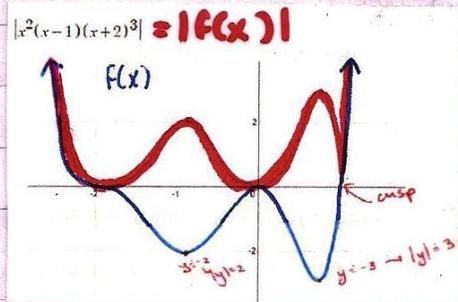
### ⑨ a) Key Information about sketching Absolute Values of Graphs

Absolute value, as a function, takes negative values and makes them positive, while already positive values remain positive.

→ On a graph, the absolute value function reflects any negative pieces of the graph that are below the x-axis in the x-axis to make them positive.

As shown in the example, all parts below the x-axis were reflected to become positive.

★ To ensure accuracy, use some critical points (turning points) to reflect the pieces of the graph. Then, sketch.

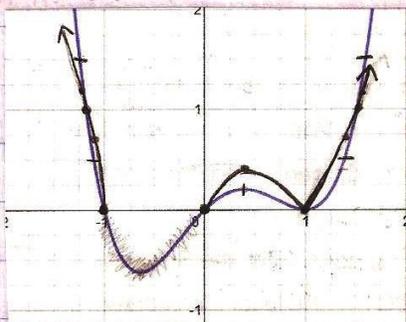


## 9c) Key information about sketching BOTH roots of polynomials

- On your graph,  $f(x)$ , find the roots of key  $y$ -values. (i.e.  $y$ -values that are max/min values, near ends of graph)
- Invariant points such as  $y=0$ ,  $y=1$ , and  $y=-1$  are called invariant because if you apply any root to them, you will get the same value. The original and even/odd root graph pass through these points.  
NOTE:  $-1$  isn't invariant for even roots!
- Even roots will have no graph for  $y$ -values  $< 0$  of the original graph

SQUARE  
ROOT  
ex.

•  $\sqrt{(x-1)^2(x)(x+1)}$



- ① Look for and plot the invariant points on your graph because they will be on the root graph.  
★ even root  $\therefore -1 \neq$  invariant

- ② Take the  $\sqrt{\quad}$  of key  $y$ -values. Remember that negative values will not exist anymore!
  - $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \approx 0.707$  ★ keep  $x$ -values the same
  - $\sqrt{1.5} \approx 1.22$
  - $\sqrt{0.20} \approx 0.44$  6 DECIMALS

- ③ Connect pieces together by looking at behaviour of how the points are plotted. For the part of the graph that doesn't exist, skip/jump over it. DON'T CONNECT!

- ④ The end behaviour of roots of polynomials is the opposite of "activity" of the end behaviour of original.

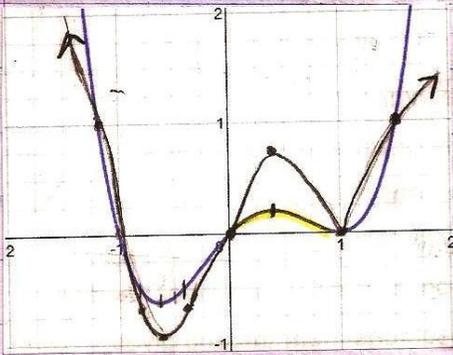
Above, from the  $x$ -intercept  $-1$ , the original concaves up (think "cup").

So, the end behaviour will concave down.

NOTICE: activity changes at  $y=1$

## ODD ROOT ex.

•  $\sqrt[3]{(x-1)^2(x)(x+1)}$



① Look for and plot the invariant points because they will be on the root graph.  $x = -1$  is also invariant because of odd root.

② Take the  $\sqrt[3]{}$  of key y-values. Take  $\sqrt[3]{}$  of negative y-values as well!

③ Connect all the pieces together by looking at the behaviour of how points are plotted.

-  $\sqrt[3]{-0.5} \approx -0.79$

-  $\sqrt[3]{-0.75} \approx -0.9$

④ The end behaviour can be determined

-  $\sqrt[3]{0.2} \approx 0.6$

the same way as it was in the previous example. Refer to step 4 for notes about change in "activity".

WHY do roots of polynomials' graphs have oval-shaped pieces?

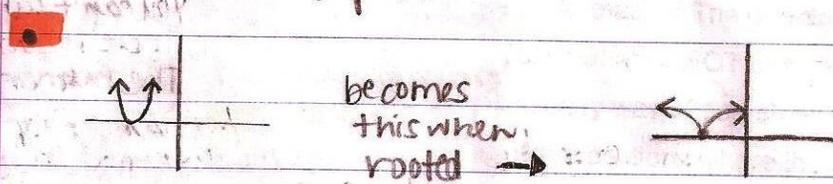
Imagine the highlighted piece above is half a circle. If you had to apply a root to this circle, since its value is below 1, the circle "piece" extends to become oval-like. Or, if your highlighted piece was originally above 1, applying a root to the value would flatten to an oval piece.

**KEY IDEA:** roots of #'s less than 1 = greater output than input

roots of #'s greater than 1 = smaller output than input.

## Characteristics of graphs of roots of polynomials

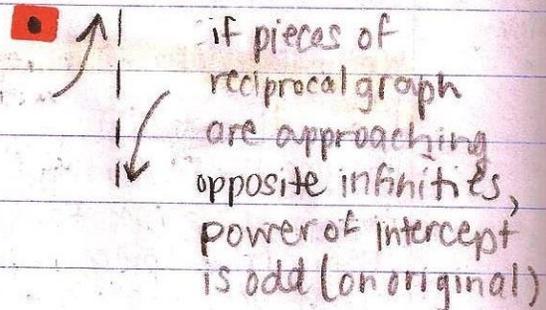
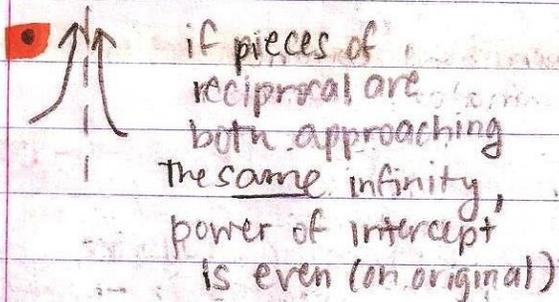
- ① If you see a bounce occurring, it CANNOT be a root graph because an intercept with multiplicity 2, when rooted, results in a sharp point.



- ② If you see gaps in a graph, think even root because graphs don't exist for negative y-values of an original polynomial.

- ③ If you see vertical asymptotes, think of reciprocal (rational). (since x intercepts  $\Rightarrow$  vertical asymptotes.)

- ④ To determine power on original x-intercept for its reciprocal graph, look at the activity along the VA.



- ⑤ When a vertical tangent is visible on a graph, think  $\sqrt[3]{\phantom{x}}$ .

Use when you have  $\frac{\infty}{\infty}$



**GROWTH RATE OF FUNCTIONS**

numerator  
denominator

fast	= $\infty$
slow	
slow	= 0
fast	

- slowest
- constant
  - logarithm
  - power ( $1/2, 2/3 \dots$ )
  - polynomial (1, 2, 3)
  - exponentials
  - factorial ( $x!$ )
  - aleph ( $x^x$ )
- fastest

This can be used to evaluate limits going to  $\pm \infty$ . ALL apply to the limits at  $+\infty$ . HOWEVER, only some apply to the limits at  $-\infty$ .

L) exponentials + logarithms: think literally! As  $x \rightarrow -\infty$ ,  $y \rightarrow 0$  in this case!

•  $\lim_{x \rightarrow \infty} \frac{3^x}{x^3}$  ① Use growth rate of functions to compare numerator / denom.

compare growths:  $\frac{3^x}{x^3} = \frac{\text{exponential}}{\text{polynomial}} = \frac{\text{faster}}{\text{slower}} = \infty$

$\therefore \lim_{x \rightarrow \infty} \frac{3^x}{x^3} = \infty$

② Write  $\therefore$  statement

•  $\lim_{x \rightarrow \infty} \frac{x^3}{3^x}$  ① Use growth rate of functions to compare numerator / denominator.

compare growths:  $\frac{x^3}{3^x} = \frac{\text{polynomial}}{\text{exponential}} = \frac{\text{slower}}{\text{faster}} = 0$

$\therefore \lim_{x \rightarrow \infty} \frac{x^3}{3^x} = 0$

② Write  $\therefore$  statement

•  $\lim_{x \rightarrow -\infty} \frac{3^x}{x^3}$

$\lim_{x \rightarrow -\infty} \frac{3^x}{x^3} = \frac{0}{-\infty} = 0$

$\therefore \lim_{x \rightarrow -\infty} \frac{3^x}{x^3} = 0$

because graph of  $3^x$  approaches zero on left because of HFA at  $y=0$

① Think of limit of graph of  $3^x$  as  $x \rightarrow -\infty$

② Think of limit of graph of  $x^3$  as  $x \rightarrow -\infty$

③ Write  $\therefore$  statement

•  $\lim_{x \rightarrow \infty} \frac{x^3}{3^x} = \frac{-\infty}{0} = -\infty(\infty)$

$= -\infty$

↑ Refer to step above ↑

$\therefore \lim_{x \rightarrow \infty} \frac{x^3}{3^x} = -\infty$

**TO GET RID OF DIVISION BY ZERO, UNDERSTAND THAT:**

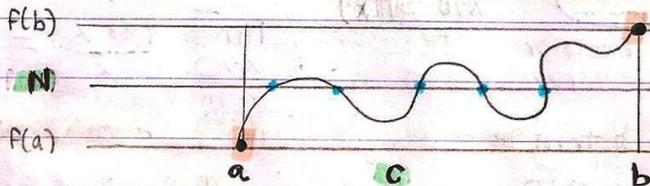
Reciprocal of 0  $\rightarrow \infty = \frac{1}{0}$

Reciprocal of  $\infty \rightarrow 0 = \frac{1}{\infty}$

$\forall$  for all  
 $\therefore$  therefore  
 $\because$  since  
 $\in$  element of

**4) Intermediate Value Theorem (IVT)**

If the function  $f$  is continuous on a closed interval  $[a, b]$  with  $N$  as any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ , then a number  $c$  in  $(a, b)$  exists such that  $f(c) = N$ . (There can be multiple but we



use this to find if there is at least ONE!

You must state all conditions of IVT !!

**FORMAT OF  $\therefore$  STATEMENT FOR USING IVT ( $N=0$  when finding roots using IVT)**

$\because f(x)$  is continuous on the closed interval  $[a, b]$  and  $f(a) < N < f(b)$   
 $\therefore$  by IVT, there exists  $c \in [a, b]$  such that  $f(c) = N$   
 or  $c \in (a, b)$

•  $2\sin(x) = 3 - 2x$  has a root in  $(0, 1)$ . verify this to be true.

$f(x) = 3 - 2x - 2\sin(x)$   $\checkmark$  continuous

$\oplus f(0) = 3 - 2(0) - 2\sin(0)$   
 $= 3 - 0 - 2(0)$   
 $= 3$

RAD MODE!

$f(1) = 3 - 2(1) - 2\sin(1)$   
 $= 3 - 2 - 2(0.84\dots)$   $\leftarrow$  STO

$\ominus \hat{=} -0.683$

① We need to find the roots to tell us if there is one or not. To do this, move all terms to one side and name the function using function notation.

② Notice that, overall, the function is continuous. The interval (open) of  $(0, 1)$  is given to us to narrow our search for a root.

③ Using the leftmost point and rightmost point, find the output values.

④ After using points of interval to find output values, we must compare them to what we are trying to find.

Our  $N$  in this example is zero because we want to see if roots  $(\neq, 0)$  exist. As long as we have a  $\oplus$  output and a  $\ominus$  output, we can apply IVT.

even though we are given  $(1, 0)$ , state it as closed here, (since  $f(x)$  is continuous)

⑤ Write conclusion invoking IVT.

$\because f(x)$  is continuous on a closed interval of  $[0, 1]$  and  $f(1) < 0 < f(0)$ ,  $\therefore$   
 by IVT, there exists  $c \in [0, 1]$  such that  $f(c) = 0$

**REALIZE:** We are looking for 2 output values where one is  $\oplus$  and  $\ominus$  to conclude that at  $c$ , the output will be the desired  $N=0$  for us

★ limits must approach 0

OH YAY, as if I missed seeing the word "trig" here it is again...

TRIG LIMITS

⑤a) KEY LIMITS

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

ULTIMATE SINE LIMIT (USL)

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{x}{\sin(x)}$$

(input and denominator must be the same here)

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

⑤b) How to evaluate a trig limit using the USL and other identities

- $\lim_{x \rightarrow 0} \frac{\cot(2x)}{\csc(x)}$  quotient ID reciprocal ID

$$= \lim_{x \rightarrow 0} \frac{\cos(2x)}{\sin(2x)} \cdot \frac{\sin(x)}{1} \cdot (2x) \quad \text{problem: } \sin(2x)$$

$$= \lim_{x \rightarrow 0} \frac{\cos(2x) \sin(x) (2x)}{(2x) \sin(2x)} \quad \text{USL}$$

$$= \lim_{x \rightarrow 0} \frac{\cos(2x) \sin(x)}{(2x)} = \lim_{x \rightarrow 0} \frac{\cos(2x)}{2} \quad \text{D.S.}$$

$$= \frac{\cos(2 \cdot 0)}{2}$$

$$= \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 0} \frac{\cot(2x)}{\csc(x)} = \frac{1}{2}$$

⑥ Now, you can do direct substitution!

⑦ Write % statement

① Make sure the limit is  $\rightarrow 0$

② Try to get everything to be sine/cosine  
ALSO NOTE WHAT THE "PROBLEM" is,

③ Think about what can be done to use the USL.

We know that on top of/below sine's input, there needs to be an identical term.

④ Multiply top/bottom by  $2x$  and get rid of factors that = 1 by USL.

⑤ Even though we got rid of our original problem, now we have another. How can we get rid of this  $x$ ? (on bottom)?  
USL again.

(6a) What is the difference between explicit and implicit equations?

**EXPLICIT EQUATION**: Shows you how to go directly from  $x$  to  $y$ .

- $y = x^3 - 3 \rightarrow$  when you know  $x$ , you can find  $y$ .

**IMPLICIT EQUATION**: doesn't directly show you how to go directly from  $x$  to  $y$ .

- $x^2 - 3xy + y^3 = 0 \rightarrow$  when you know  $x$ , how do you find  $y$ ???

With implicit equations, it may be hard (or impossible) to go to  $y$  directly from  $x$ . Also, the equation may not be a function!

EVERY TIME YOU SEE A Y, THINK THAT AN X IS HIDDEN INSIDE OF IT. (y on its own is a function)

(6b) Implicit Differentiation using power/chain rule

IN PRIME NOTATION

•  $y^3 + y^2 - 6y + x^2 = -8$  Find  $\frac{dy}{dx}$

$\frac{d}{dx} [y^3 + y^2 - 6y + x^2 = -8]$

$\frac{d}{dx} y^3 + \frac{d}{dx} y^2 - 6 \frac{d}{dx} y + \frac{d}{dx} x^2 = \frac{d}{dx} -8$

$3(y)^2(y') + 2(y)(y') - 6(y') + 2x = 0$

since y is a function

$3y^2 y' + 2y y' - 6y' = -2x$

$y'(3y^2 + 2y - 6) = -2x$

$y' = \frac{-2x}{(3y^2 + 2y - 6)}$

① State what you are doing.

\* cannot say  $\frac{dy}{dx}$  since we have

an equation (not with output y) to differentiate.

② Take the derivative of each term on each side!

③ Since asked to find  $\frac{dy}{dx}$  (or  $y'$ ), isolate for  $y'$ .

More  $2x$  then common factor.

IN LEIBNIZ NOTATION

•  $\sin x + \sin(2y) = 1$  Find  $\frac{dy}{dx}$

$\frac{d}{dx} [\sin x + \sin(2y) = 1]$

$\frac{d}{dx} \sin x + \frac{d}{dx} \sin(2y) = \frac{d}{dx} 1$

$\cos x + \cos(2y) \left( 2 \frac{d}{dx} y \right) = 0$

← applied constant derivative rule.

$\cos x + 2 \cos(2y) \frac{dy}{dx} = 0$

more  $\cos x$  divide  $2 \cos(2y)$

$\frac{dy}{dx} = \frac{-\cos x}{2 \cos(2y)}$

③ Since asked to find  $\frac{dy}{dx}$ , isolate.

EXTRA:  $\frac{d^2 y}{dx^2} = y''$  (How?)

$\frac{d}{dx} \left( \frac{d}{dx} [y] \right)$

$= \frac{d}{dx} \left( \frac{d}{dx} \right) (y)$

$= \frac{d^2}{(dx)^2} y = \frac{d^2 y}{dx^2}$

NOTE: d (top) alone dx (bottom) ONE UNIT

⑥ Implicit Differentiation using product/quotient rule

PRODUCT  
RULE

•  $xy = 3x^2$  find  $\frac{dy}{dx}$

① state what you are doing

LEIBNIZ  
NOTATION

$$\frac{d}{dx} [xy = 3x^2]$$

← constant derivative rule.

② Take the derivative  
★ APPLY PRODUCT  
RULE ★

$$(x) \frac{d}{dx} (y) + (y) \frac{d}{dx} (x) = 3(2x)$$

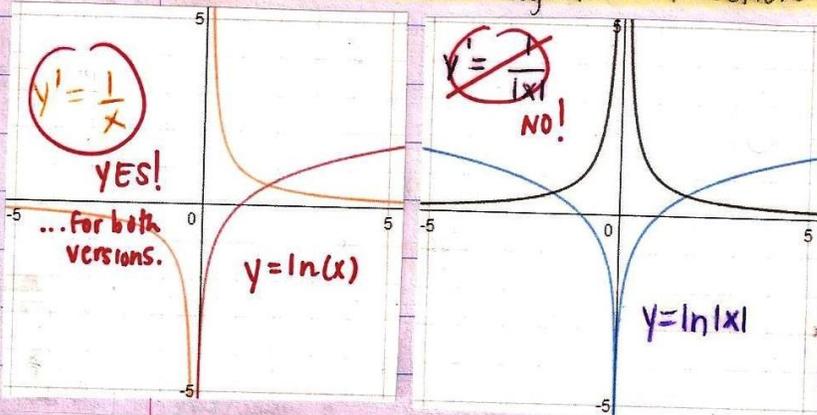
$$(x) \left( \frac{dy}{dx} \right) + (y) (1) = 6x$$

③ since we are asked to find  $\frac{dy}{dx}$   
ISOLATE.

move y  
divide x

$$\frac{dy}{dx} = \frac{6x - y}{x}$$

Qc) Absolute value inside a logarithm function... *does it matter?*

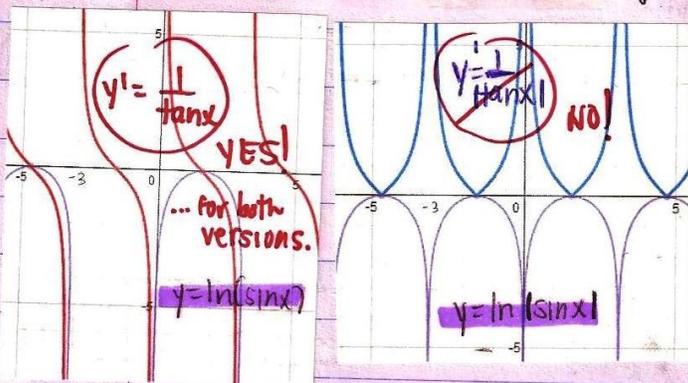


i) When absolute value is applied to the input of a logarithm, the domain becomes  $x \in \mathbb{R}$ .

The red graph is the original with its derivative graphed. We know that all inputs of  $\ln(x)$  should be greater than zero. So, technically, only the RS of the rational is graphically need to represent the derivative.

The blue graph has an absolute value input. This extends the domain. If we look at the derivative  $\frac{1}{x}$  with an absolute value on  $x$ , it no longer reflects  $\ln|x|$ .

Slopes on left SHOULD be close to zero then (as they  $\rightarrow x=0$ ) slopes should be large and negative. Only the graph of  $\frac{1}{x}$  reflects this.



ii) You should not record absolute value on the derivative even if the original input had it because: on input  $\rightarrow$  domain changes, on derivative  $\rightarrow$  outputs become  $\ominus$  and no longer represent correct slopes.

The purple graph on the left is the original with its derivative graphed. Since the domain of  $\ln$  must be positive, only positive outputs for  $\sin(x)$  are used as inputs. So, not all of the derivative graph is needed to represent the derivative of  $y = \ln(\sin(x))$  graphically. The absolute value

on the input of the second purple graph makes the domain  $x \in \mathbb{R}$  since now, all  $\sin(x)$  outputs will be positive! The absolute value on  $\tan(x)$  makes all outputs of  $\frac{1}{\tan(x)}$  positive so it doesn't reflect slopes of  $y$  correctly.

•  $y = x^x$

①  $\ln(y) = \ln(x^x)$

↑  
bring power  
down

②

$$\ln(y) = (x)(\ln x)$$

③  $\frac{1}{(y)(\ln e)} \cdot y' = (x) \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(x)$

↑ since  $\log_e(e)$

$$\frac{y'}{y} = (x) \left[ \frac{1}{(x)(1)} \cdot (1) \right] + (\ln x) (1)$$

$$\frac{y'}{y} = 1 + \ln(x)$$

$$y' = [1 + \ln(x)](y) \quad ④$$

$$y' = [1 + \ln(x)](x^x)$$

① This is where  $x$  is in both the base and the exponent.

### WE MUST USE LOGARITHMIC DIFFERENTIATION.

① → Take log or  $\ln$  of both sides and the whole side

② → Simplify using log laws

③ → Take derivative implicitly

④ → Isolate for  $y'$  then replace all  $y$ 's with expression containing  $x$ 's.

ⓑ)

### Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

ⓐ) Proof of  $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

$y = \cos^{-1} x$

① Write out the original function

$\cos(y) = \cos(\cos^{-1}(x))$

$\cos y = x$

$\sin^{-1}(\sin(\pi/2 - y)) = \sin^{-1}(x)$

② We want y alone. So, take "sin<sup>-1</sup>" of both sides.

② Apply "cos" to both sides to get x alone.

$\pi/2 - y = \sin^{-1} x$

③ Use two definitions of y and make Ls = Rs

③ Use cofunction ID of cosine.

$\pi/2 - \sin^{-1} x = y$

④ take  $\frac{d}{dx}$  of both sides and use definition of  $\sin^{-1} x$ .

$\pi/2 - \sin^{-1} x = \cos^{-1} x$

$\frac{d}{dx}(\pi/2 - \sin^{-1} x) = \frac{d}{dx} \cos^{-1} x$

$0 - \frac{1}{\sqrt{1-x^2}} = \frac{d}{dx} \cos^{-1} x$

$\frac{-1}{\sqrt{1-x^2}} = \frac{d}{dx} \cos^{-1} x$

Proof of  $\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2}$       $y = \cot^{-1}(x)$

① Write out the original function

$$\cot(y) = \cot(\cot^{-1}(x))$$

② Apply "cot" to both sides to get x alone.

$$\cot y = x$$

④ Take "tan<sup>-1</sup>" of both sides to get y alone.

$$\tan^{-1}(\tan(\frac{\pi}{2} - y)) = \tan^{-1}x$$

③ Use cofunction ID of cotangent

$$\frac{\pi}{2} - y = \tan^{-1}x$$

⑤ Use the two definitions of y to make LS=RS

$$\frac{\pi}{2} - \tan^{-1}x = y$$

⑥ Take  $\frac{d}{dx}$  of both sides and use definition of  $\tan^{-1}x$ .

$$\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$$

$$\frac{d}{dx}(\frac{\pi}{2} - \tan^{-1}x) = \frac{d}{dx}(\cot^{-1}x)$$

$$0 - \frac{1}{1+x^2} = \frac{d}{dx} \cot^{-1}x$$

$$\boxed{\frac{-1}{1+x^2} = \frac{d}{dx} \cot^{-1}(x)}$$

### IMPORTANT NOTE ABOUT USING DERIVATIVE OF INVERSE TRIG FUNCTION FORMULAS:

- Whatever the entire input is, that is what is inserted in place of every x in the general formula.
- Whenever the input is not only x, you must take the derivative of the input and multiply it by the general formula.

#### GENERAL steps on finding derivative of inverse trig functions

- ① Apply primary/secondary trig ratios to get rid of inverse
- ② Draw picture at this point
- ③ Do implicit differentiation
- ④ Use picture to sub in ratios for other trig functions while trying to isolate y'
- ⑤ Simplify

## 06) EVT - Extreme Value Theorem

If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c$  and  $d$  in  $[a, b]$ .

### How to use EVT to find absolute max/min

- 1) Find critical points ...  $c \in [a, b]$ . Do this by taking derivative, and making it equal to zero.

NOTE: critical number is  $\rightarrow f'(c) = 0 = t, p, \text{ saddle pt}$  ← numerator

$\rightarrow f'(c) = \text{DNE} = \text{cusp, corner, hole, jump...}$

↑ denominator

- 2) Find  $f(a)$ ,  $f(b)$ , and all of  $f(c)$  since multiple critical pts may exist.

- 3) Compare all results and pick out largest (abs MAX) and smallest (abs MIN)

Sometimes, you can't evaluate a critical point, so, use a limit to evaluate value as function approaches it.

### (10) Mean Value Theorem (MVT)

If  $f$  is a function that is continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$  (so derivative @  $a$  or  $b$  doesn't have to exist) there exists a number  $c$  in  $(a, b)$  such that:  $f'(c) = \frac{f(b) - f(a)}{b - a} = \text{slope on } [a, b]$ .

Or equivalently:  $f'(c)(b - a) = f(b) - f(a)$

- Verify that  $f(x) = x^3 + x - 1$  satisfies the properties of MVT on  $[0, 2]$  and then find the numbers that satisfy MVT.

since both satisfy  $\exists x \in \mathbb{R}$ ?  
they satisfy  $[0, 2]$

$f(x)$  is a polynomial  $\therefore$  continuous for  $\exists x \in \mathbb{R}$ ?  
 $f'(x) = 3x^2 + 1$   $\therefore$  derivative exists for  $\exists x \in \mathbb{R}$ ?

- Verify that  $f(x)$  is continuous on  $[a, b]$
- Show that it is differentiable on  $(a, b) \rightarrow$  Do this by finding the derivative and showing derivative is true for  $x \in \mathbb{R}$  or at least  $[a, b]$

Apply MVT: There exists  $c \in (a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

④ Evaluate to find  $c$

③ Apply MVT

using def. of  $f'$  and  $c$  as a constant)

$$f'(c) = 3c^2 + 1$$

⑤ Make L.S = R.S

⑥ Write  $\therefore$  statement

$$3c^2 + 1 = \frac{f(2) - f(0)}{2 - 0}$$

$$\frac{3c^2}{3} = \frac{4}{3}$$

$\therefore$  at points  $x = \pm \frac{2}{\sqrt{3}}$

$$3c^2 + 1 = \frac{9 - (-1)}{2 - 0}$$

$$\sqrt{c^2} = \sqrt{\frac{4}{3}}$$

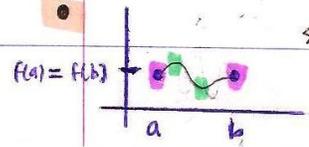
we have the same average slope as the average slope (slope) on  $[0, 2]$ .

$$c = \pm \frac{2}{\sqrt{3}}$$

### Rolle's Theorem

This is a special case of MVT where, if the outputs on both numbers of the closed interval are the same, MVT can still be applied but we can go further to say that there exists some number where the derivative is 0.

If  $f$  is a function where MVT can be applied, AND  $f(a) = f(b)$  then there exists some number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .



★ The graph in between  $a$  and  $b$  can be drawn in many ways but there MUST always be at least one point where derivative  $f'(c) = 0$